Riemannian Geometry

MAS621

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1 Tangent Space

Definition 1. A topological space is an n**-dimensional manifold** if locally $\mathbb{R}^n$ Hausdorff second countable space. For a manifold M, $\dim M$ denotes n, the dimension of M, and (U, φ) is a **chart** of M if $U \subseteq M$ is an open set and $\varphi : U \to \mathbb{R}^n$ is an embedding. For two charts $(U, \varphi), (V, \psi)$ with $U \cap V \neq \emptyset$, a **transition map between** φ **and** ψ is the map $\psi \circ \varphi^{-1}|_{\varphi(U\cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^n \to \psi(U \cap V) \subseteq \mathbb{R}^n$. A manifold with given charts is **smooth** if every transition map is smooth.

Remark. By definition, locally $\mathbb{R}^n$, of manifold M, for any open subset U of M and $p \in U$, there exists a chart (V, φ) such that $p \in V \subseteq U$.

Remark. Inverse map of a transition map is also a transition map, so indeed, every transition map of a smooth manifold is a diffeomorphism to its image.

Remark. A set of charts $\{(U_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ is an atlas if U_α covers whole manifold. Each smooth atlas can be extended to a unique maximal smooth atlas, so we may say a smooth atlas determines whole smooth structure of the manifold. Because of some technical reason, we always consider the maximal smooth structure that could be given for a manifold, which means, if (U, φ) does not disturb smoothness of the given manifold, then it would be considered as a chart of the manifold. For example, $\mathbb{R}^n$ itself has a smooth atlas $(\mathbb{R}^n, \mathrm{id})$ and without special mention, $\mathbb{R}^n$ is considered as a smooth manifold with the maximal smooth atlas generated from this atlas, so (U, φ) with open subset U of $\mathbb{R}^n$ and $\varphi : U \to \mathbb{R}^n$ which is diffeomorphism between U and $\varphi(U)$ is always a chart of $\mathbb{R}^n$.

Definition 2. A map $f : M \to N$ between smooth manifolds M, N is smooth at $p \in M$ if f is continuous at p and there exists a chart (U, φ) of M where $p \in U$ and (V, ψ) of N where $f(p) \in V$ such that $\psi \circ f \circ \varphi^{-1}|_{\varphi(f^{-1}(V)\cap U)} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth at $\varphi(p)$. A map $f : M \to N$ is smooth if smooth at every point.

Remark. If a map $f : M \to N$ is smooth at p, then for any charts $(U, \varphi), (V, \psi)$ where $p \in U \subseteq M$ and $f(p) \in V \subseteq N$, $\psi \circ f \circ \varphi^{-1}|_{\varphi(f^{-1}(V)\cap U)}$ is smooth at $\varphi(p)$. So, to check smoothness of a map with generating atlas of manifolds are given, we only need to check from charts in atlases.

Remark. Smoothness of a map $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ between in real analytic sense and in manifold sense coincide.

Remark. This is equivalent to say there exists a chart (U, φ) of M and (V, ψ) of N such that $p \in U$, $f(U) \subseteq V$ and $\psi \circ f \circ \varphi^{-1}$ is smooth.

Remark. Easily, composition of smooth maps is smooth.

Definition 3. For a manifold M, $C^\infty(M)$ is the set of smooth maps from M to $\mathbb{R}$, and $C^\infty(M, p)$ is the set of maps from M to $\mathbb{R}$ which is smooth at p.

Remark. $C^\infty(M)$ and $C^\infty(M, p)$ are vector spaces naturally.

Definition 4. A **smooth curve** on a manifold M is a smooth map from an open interval to M.

Definition 5. A **tangent vector to a curve** c is the operator based on a smooth curve c on M where 0 is in the domain of c, which is

$$c'(0) : C^\infty(M, c(0)) \to \mathbb{R}$$

defined as $c'(0)(f) = c'(0)f = \frac{d}{dt}(f \circ c)(0)$ where t is the parameter for c. A **tangent vector at** p is a tangent vector to a smooth curve c where $c(0) = p$. T_pM is the set of tangent vectors at p which is called as the **tangent space** of M at p.

Remark. A tangent vector is always a linear operator.

Proposition 6. A tangent space of a smooth manifold M is an $\mathbb{R}$-vector space of dimension $\dim M$.

Proof. Fix $p \in M$ and chart (U, φ) where $p \in U$. Let $n = \dim M$. For any smooth curve $c : I \to M$ where $c(0) = p$ and $f \in C^\infty(M, p)$, let $\tilde{c} = \varphi \circ c|_{I\cap c^{-1}(U)} : I \cap c^{-1}(U) \to \mathbb{R}^n$ and $\tilde{f} = f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^n \to \mathbb{R}$. Denote $\tilde{c}(t) = (x_1(t), \cdots, x_n(t))$. Since f, c are smooth maps, so $\tilde{f}, \tilde{c}$ are smooth maps in analytic sense. Now, $f \circ c|_{I\cap c^{-1}(U)} = f \circ \varphi^{-1} \circ \varphi \circ c|_{I\cap c^{-1}(U)} = \tilde{f} \circ \tilde{c}$, so $\frac{d}{dt} f \circ c(0) = \frac{d}{dt} \tilde{f} \circ \tilde{c}(0) = \frac{d}{dt} \tilde{f}(x_1(t), \cdots, x_n(t))|_{t=0} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \tilde{f}(\varphi(p)) x_i'(0) = \sum_{i=1}^n x_i'(0) \frac{\partial}{\partial x_i}|_{\varphi(p)} \tilde{f}$. Now, let $L_\varphi : T_pM \to \mathbb{R}^n$ be $L_\varphi(c'(0)) = (x_1'(0), \cdots, x_n'(0))$. Then, summation formula of $\frac{d}{dt} f \circ c(0)$ proves L_φ is injective. Moreover, for given $(v_1, \cdots, v_n)$, choose $\epsilon > 0$ satisfying $\varphi(p) + \prod_{i=1}^n [-\epsilon v_i, \epsilon v_i] \subseteq \varphi(U)$ which is always possible, and define $c : (-\epsilon, \epsilon) \to M$ as $c(t) = \varphi^{-1}((tv_1, tv_2, \cdots, tv_n) + \varphi(p))$. Then, $c(0) = p$ and $x_i'(0) = v_i$, so $L_\varphi(c'(0)) = (v_1, \cdots, v_n)$ which proves L_φ is surjective. This defines a vector space structure for T_pM. Since differential of transition map is a bijective linear function, so vector space structure defined respectively to φ is equal to vector space structure defined based on different charts. Thus, the vector space structure for T_pM is well-defined, which means T_pM is an $\mathbb{R}$-vector space of dimension $\dim M$. □

Remark. In the formula $L_\varphi^{-1}(x_1'(0),\cdots,x_n'(0))f = c'(0)f = \sum_{i=1}^n x_i'(0)\frac{\partial}{\partial x_i}|_{\varphi(p)}(f\circ\varphi^{-1})$, $\frac{\partial}{\partial x_i}$ seems like a basis just in symbol, where $c'(0)$ could be thought as $\sum_{i=1}^n x_i'(0)\frac{\partial}{\partial x_i}|_{\varphi(p)}$. So, we denote element of T_pM as $\sum_{i=1}^n v_i\frac{\partial}{\partial x_i}|_p$ where $\frac{\partial}{\partial x_i}|_p$ is a tangent vector such that $\left(\frac{\partial}{\partial x_i}|_p\right)f = \frac{\partial}{\partial x_i}|_{\varphi(p)}(f\circ\varphi^{-1})$ with hidden chart map $\varphi(p) = (x_1(p),\cdots,x_n(p))$.

Definition 7. For a manifold M, the **tangent bundle** of M, TM, is $\bigcup_{p\in M}\{p\}\times T_pM$. The natural projection from TM to M is $\pi : TM \to M$ such that $\pi(p,v) = p$.

Remark. TM has a natural manifold structure, where $\dim TM = 2\dim M$.

Definition 8. For a map $F : M \to N$ between smooth manifolds which is smooth at $p \in M$, **differential of** F **at** p is a linear map $dF_p : T_pM \to T_{F(p)}N$ such that $dF_p(v)f = v(f\circ F)$ for every $f \in C^\infty(N, F(p))$, which is well-defined since for $v = c'(0)$, $v(f\circ F) = \frac{d}{dt}(f\circ F\circ c)(0) = (F\circ c)'(0)f$, which means $dF_p(v) = (F\circ c)'(0)$.

Remark. For a smooth curve defined on 0, $dc_0(\frac{d}{dt})f = \frac{d}{dt}(f\circ c)(0) = c'(0)f$. Thus, $c'(0) = dc_0\frac{d}{dt}$. Similarily, $c'(t_0)$ is defined as $c'(t_0)f = dc_{t_0}(\frac{d}{dt})f = \frac{d}{dt}(f\circ c)(t_0)$, where $c'(t_0) \in T_{c(t_0)}M$.

Remark. If $F : L \to M$ and $G : M \to N$, $d(G\circ F)_p(v) = (G\circ F\circ c)'(0) = dG_{F(c(0))}((F\circ c)'(0)) = dG_{F(p)}(dF_p(v))$.

Definition 9. A **vector field** X on a smooth manifold M is a map from M to TM such that $X(p) \in \{p\}\times T_pM$. A vector field is smooth if it is smooth as map between manifolds. Also, we may consider $X(p) \in T_pM$ since $X(M) \subseteq TM$ could be understood as a function from M to $\bigcup_{p\in M} T_pM$ and then, $X(M)(p) \in T_pM$. To make this notation be more natural, TM could be understood as disjoint union of T_pMs. We will use both conventions freely.

Remark. For $v \in T_pM$, we denote $v = \sum_{i=1}^n v_i\frac{\partial}{\partial x_i}|_p$. Then, vector field X, $X(p)$ can be denoted as $X(p) = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}|_p$ where a_i are functions to $\mathbb{R}$. In a fixed hidden chart, smoothness of X is obtained by smoothness of a_is and notation $X = \sum_{i=1}^n a_i\frac{\partial}{\partial x_i}$ might be available.

Remark. For each $f \in C^\infty(M)$ and smooth vector field X, we may think $Xf \in C^\infty(M)$ as $Xf(p) = X(p)f$. In this point of view, X also be understood as a map $C^\infty(M) \to C^\infty(M)$.

For smooth vector fields X, Y, notation YXf makes sense for smooth map f. But YX might not be a vector field since every vector field satisfies $X(fg) = (Xf)g + f(Xg)$ but $(YX)(fg) = Y(X(fg)) = Y((Xf)g + f(Xg)) = (YXf)g + (Xf)(Yg) + (Yf)(Xg) + f(YXg)$ which might be different to $(YXf)g + f(YXg)$.

Definition 10. For smooth vector fields X, Y, **Lie derivative of** X **and** Y is $[X,Y] = XY - YX$.

Remark. By simple calculation, $[X,Y](fg) = ([X,Y]f)g + f([X,Y]g)$, so it works like a vector field. Actually, for $X = \sum_{i=1}^n X^i\frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n Y^i\frac{\partial}{\partial x_i}$, $[X,Y] = \sum_{i=1}^n (XY^i - YX^i)\frac{\partial}{\partial x_i}$, so it is really a vector field.

1.1 Connection

Definition 11. For a smooth manifold M, $\Gamma^k(TM)$ is the set of C^k vector fields on M.

Definition 12. For a smooth manifold M, ∇ is a **connection of** M if it is a map $\nabla : TM \times \Gamma^1(TM) \to TM$ such that

1. $\nabla((p,v), Y) \in T_pM$.
2. For every $\alpha, \beta \in \mathbb{R}, \xi, \eta \in T_pM$, $\nabla(\alpha\xi + \beta\eta, Y) = \alpha\nabla(\xi, Y) + \beta\nabla(\eta, Y)$.
3. $\nabla(\xi, Y_1 + Y_2) = \nabla(\xi, Y_1) + \nabla(\xi, Y_2)$.
4. For $f \in C^\infty(M)$, $\nabla((p,v), fY) = (vf)Y(p) + f(p)\nabla((p,v), Y)$

Remark. If f is a constant function, $vf = 0$, so $\nabla(\xi, \alpha Y_1 + \beta Y_2) = \alpha\nabla(\xi, Y_1) + \beta\nabla(\xi, Y_2)$ is also satisfied.

For a connection, $\nabla_\xi Y = \nabla(\xi, Y)$ is also used as notation. If a connection is defined, then for any smooth vector fields X, Y, $\nabla_X Y$ is also a vector field defined as $(\nabla_X Y)(p) = \nabla_{X(p)}Y$, so ∇ can be understood as map $\Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma(TM)$. Connection is **smooth** if its image is in $\Gamma^\infty(TM)$.

Proposition 13. For a connection ∇, $\nabla_{(p,v)}Y$ is determined by the restriction of Y to any open neighborhood U of p.

Proof. It is enought to show $Y|_U = 0$ implies $\nabla_\xi Y = 0$. Suppose $Y|_U = 0$ and let f be a bump function, where $f : M \to \mathbb{R}$ satisfying $f(p) = 0$ and $f|_{M\setminus U} \equiv 1$. Then, $fY = Y$. Thus, $\nabla_\xi Y = \nabla_\xi fY = (\xi f)Y|_p + f(p)\nabla_\xi Y = 0$. ∎

Remark. Conversely, for a C^1 vector field Y only defined on U, for each $p \in U$, we may consider a bump function $\varphi : M \to \mathbb{R}$ such that support of φ is included in U and $\varphi|_V \equiv 1$ where V is an open subset of U containing p. Then, define $\overline{Y} = \begin{cases} \varphi Y & \text{in } U \\ 0 & \text{o.w.} \end{cases}$. Then, $\overline{Y} \in \Gamma^1(TM)$ and $\nabla_{(p,v)}\overline{Y}$ only depends on $\overline{Y}|_V = Y|_V$ which is independent to choice of the bump function. Thus, define $\nabla_{(p,v)}Y = \nabla_{(p,v)}\overline{Y}$ is well-defined.

Now, to calculate $\nabla_\xi Y$ for $\xi \in T_pM$, choose a chart $x : U \to \mathbb{R}$ with $p \in U$. Denote $\frac{\partial}{\partial x_i}$ as ∂_i. Then, $\xi = \sum_j \xi^j \partial_j|_p$. Define **Christoffel symbols** as Γ^l_{jk} which satisfying $\nabla_{\partial k}\partial j = \sum_l \Gamma^l_{jk}\partial_l$ on U. Then, if $Y = \sum_j \eta^j \partial_j$ on U, we finally get

$$\begin{aligned}
\nabla_\xi Y &= \sum_k \xi^k \nabla_{\partial_k|_p} Y = \sum_k \xi^k \nabla_{\partial_k|_p} \sum_j \eta^j \partial_j = \sum_k \sum_j \xi^k((\partial_k|_p \eta^j)\partial_j|_p + \eta^j(p)\nabla_{\partial_k|_p}\partial_j) \\
&= \sum_k \left(\sum_l \xi^k(\partial_k|_p\eta^l)\partial_l|_p + \xi^k \sum_{j,l} \eta^j(p)\Gamma^l_{jk}(p)\partial_l|_p \right) \\
&= \sum_l \left(\sum_k \xi^k(\partial_k|_p\eta^l) + \sum_{j,k} \xi^k \eta^j(p)\Gamma^l_{jk}(p) \right) \partial_l|_p
\end{aligned}$$

In other word, Christoffel symbols determine the connection. Moreover, any choice of Christoffel symbol and above calculation as definition, we can define a local connection always. Using a partition of unity, those local connections can be combined into a global connection. Lastly, using above formula, we can calculate connection if smooth curve c satisfying $c(0) = p, c'(0) = \xi$ and $Y \circ c$ is given since $\sum_k \xi^k(\partial_k|_p\eta^l) = \xi\eta^l = (\eta^l \circ c)'(0)$.

1.2 Parallel Transportaion of vector field

Definition 14. For a path $\omega : (a,b) \to M$, a **vector field along a path** ω is a map $X : (a,b) \to TM$ such that $\pi \circ X = \omega$ where π is the natural projection $TM \to M$. In other words, $X(t) \in T_{\omega(t)}M$.

Remark. If a chart $x : U \to \mathbb{R}^n$ is given, then we may denote $\omega^j = x^j \circ \omega$ with appropriate domain reduction and $X = \sum_j \xi^j(\partial_j \circ \omega)$ where $\xi^j : (a,b) \to \mathbb{R}$.

Definition 15. If $X = \sum_j \xi^j(\partial_j \circ \omega)$ is a given vector field along a path ω with a chart, then for given Christoffel symbols in the chart, i.e., given connection on M, $\nabla_t X$ is a vector field along ω defined as

$$\nabla_t X = \sum_l \left((\xi^l)' + \sum_{j,k} (\Gamma^l_{jk} \circ \omega)\xi^j(\omega^k)' \right) (\partial_l \circ \omega)$$

Remark. It can be done, with change of variable among Christoffel symbols, that above definition is independent to choice of a chart, so is well-defined.

Proposition 16. For vector fields X, Y along a path $\omega : (a,b) \to M$ and $f : (a,b) \to \mathbb{R}$, followings are true.

1. $\nabla_t(X + Y) = \nabla_t X + \nabla_t Y$
2. $\nabla_t(fX) = f'X + f\nabla_t X$

Proof. Omit.

Remark. Actually, for a vector field X, $\nabla_t(X \circ \omega) = \nabla_{\omega'(t)}X$.

Definition 17. For smooth manifolds M, N and a map $\phi : N \to M$, X is a **vector field along a map** ϕ if $X : N \to TM$ satisfies $\pi \circ X = \phi$, which means $X(p) \in T_{\phi(p)}M$.

Proposition 18. For a differentiable vector field X along $\phi : N \to M$, fix $q \in N$, $\xi \in T_qN$ and ∇ on M. Then, for any curve $\omega : I \to N$ with $\omega(0) = q, \omega'(0) = \xi$, $\nabla_t(X \circ \omega)(0)$ is unique, where ∇_t is calculated along $\phi \circ \omega$.

Proof. Omit.

Definition 19. For smooth manifolds M, N and a vector field X along a map $\phi : N \to M$, the **derivative of** X **along** ϕ **in the direction** $\xi \in T_qN$ with given ∇ on M is $\nabla_\xi X = \nabla_t(X \circ \omega)(0)$ where ω is a smooth curve on N satisfying $\omega(0) = q, \omega'(0) = \xi$.

Remark. Even though $\phi = \mathrm{id}_M$, meaning of two ∇_ξ coincide, so this has no problem. Moreover, for a given M and a path ω, if we choose $N = \mathbb{R}$, $\phi = \omega$, we may conclude that $\nabla_t = \nabla_{\partial_t}$. Moreover, $\nabla_\xi(X \circ \phi) = \nabla_{d\phi_q(\xi)}X$ in general.

1.3 Parallel Translation of vector fields

Definition 20. A vector field X along a path ω on a smooth manifold M with a given connection is **parallel** if $\nabla_t X \equiv 0$.

Proposition 21. For a given path ω on a smooth manifold M with a given connection ∇, for every t_0 in domain of ω and $\xi \in T_{\omega(t_0)}M$, there exists a unique vector field X along ω such that $X(t_0) = \xi$ and X is parallel.

Proof. Omit. It is from the uniqueness of solution of linear ODE with given initial condition. Observe $\nabla_t X = 0$ is exactly equivelent to a first order linear ODE with n unknown functions $\xi^1, \cdots, \xi^n$.

Remark. It means the dimension of the space of parallel vector fields along a curve is exactly the dimension of the tangent space, which is equal to the dimension of the manifold.

Definition 22. For a smooth manifold M with ∇ and a path ω on M, the **parallel translation from $T_{\omega(t_1)}M$ to $T_{\omega(t_2)}M$** is the linear map denoted as τ_{t_1,t_2} which satisfies $\tau_{t_1,t_2}(\xi) = X_\xi(t_2)$ where X_ξ is the unique parallel vector field along ω with $X_\xi(t_1) = \xi$. Note that linearity is obtained from the linearity of ODE.

Remark. Parallel translation along curve is **dependent** to choice of ω. Observe that τ_{t_1,t_2} and τ_{t_2,t_1} are inverse to each other, so they are all isomorphism, which means translating vector field along path does not vanish if initial is nonzero. Thus, if not dependent, we always be available to construct well-defined nonvanishing vector field for every smooth manifold, which is impossible by Borsuk-Ulam theorem. Also, for S^2, using parallel translation along triangle, which is piecewise smooth, that be different from original vector easily.

Theorem 23.

$$\nabla_t X(t_0) = \lim_{t \to t_0} \frac{\tau_{t,t_0}(X(t)) - X(t_0)}{t - t_0}$$

Proof. Omit. Choose linealy independent parallel vector fields E_j and denote $X(t) = \sum_j \xi^j(t) E_j(t)$. Compare both side.

1.4 Flows and integral curves

Definition 24. For a given vector field V of a smooth manifold M, a curve $\gamma : I \to M$ is an **integral curve of V** if $\gamma'(t) = V|_{\gamma(t)}$

Remark. For a chart (U, φ), $\tilde{\gamma} = \varphi \circ \gamma$ and $\tilde{Y} = d\varphi \circ V$, γ is an integral curve if and only if $\tilde{\gamma}'(t) = \tilde{Y}(\gamma(t))$, which introduce an ODE.

Theorem 25. For a given vector field V of a smooth manifold M, for every point $p \in M$, there exists a neighborhood W of p, an interval $I = (-\epsilon, \epsilon)$ and a map $F : W \times I \to M$ such that

1. For every $q \in W$, $F(q, -)$ is an integral curve of V at q, which means $F(q, 0) = q$.
2. F is differentiable.

Proof. Omit. It is also from the ODE theory.

Remark. This F satisfies $F(F(q,t), s) = F(q, t+s)$, which is the property that defines a local flow.

Remark. Moreover, for a given vector field V, F can be extended to a unique maximal flow $F^* : \bigcup_p \{p\} \times I_p \to M$ such that I_p is an open interval containing 0 and if $s \in I_p$ then $I_{F^*(p,s)} = I_p - s$. In this maximal flow, $F^*(p, -)$ is the unique maximal integral curve of V starting at p, which means $F^*(p, 0) = p$.

1.5 Geodesics and geodesic flows

Any differentiable curve $\gamma : I \to M$ of a smooth manifold M can be extended to a curve of TM as $(\gamma(t), \gamma'(t))$ naturally. Moreover, natural charts of TM is for any chart (U, x), $(\pi^{-1}(U), Q)$ where $Q : \pi^{-1}(U) \to \mathbb{R}^{2n}$ such that $Q(\xi) = (q(\xi), \dot{q}(\xi))$ with $q = x \circ \pi$ and $\dot{q}(\xi) = \xi x$, where this notation means just applying since each tangent vector maps $\mathbb{R}$ -valued functions to $\mathbb{R}$ value, so it also can be thought as mapping from $\mathbb{R}^n$-valued functions to $\mathbb{R}^n$ value, just apply componentwisely.

Definition 26. A C^l path $\omega : (a, b) \to M$ with $l \geq 2$ for a smooth manifold M with a connection is a **geodesic** if $\nabla_t \omega' = 0$. i.e., ω' is parallel along ω.

By **Definition 15.**, a geodesic is a curve $\omega : (a, b) \to M$ which satisfying following system of ODEs.

$$\omega^{l''} + \sum_{j,k} (\Gamma^l_{jk} \circ \omega) \omega^{j'} \omega^{k'} = 0$$

for $l = 1, 2, \cdots, \dim M$, which is 2nd order. We may reduce the order of the equation as

$$\begin{aligned} \omega^{l'} &= y^l \\ y^{l'} &= -\sum_{j,k} (\Gamma^l_{jk} \circ \omega) y^j y^k \end{aligned}$$

Generally, for $Q : \pi^{-1}(U) \to \mathbb{R}^{2\dim M}$, a natural chart of TM, by definition, $\xi = \sum_j \dot{q}^j(\xi)\partial_j|_{\pi(\xi)}$. Then, if we consider ω as a smooth curve over TM, as $(\omega(t), y(t))$, above equation means it becomes an integral curve of the vector field

$$\mathcal{G} = \sum_l (\dot{q}^l \frac{\partial}{\partial q^l} - \sum_{j,k} (\Gamma^l_{jk} \circ \pi) \dot{q}^j \dot{q}^k \frac{\partial}{\partial \dot{q}^l})$$

Definition 27. The unique maximal flow of $\mathcal{G}$ is called as the **geodesic flow**.

Proposition 28. For the geodesic flow φ, define $\gamma_\xi : I_\xi \to M$ as $\gamma_\xi(t) = \pi(\varphi(\xi, t))$. Then, γ_ξ is the unique maximal geodesic in M such that $\gamma_\xi(0) = \pi(\xi)$ and $\gamma'_\xi(0) = \xi$.

Remark. By its definition, $\gamma'_\xi(t) = \varphi(\xi, t)$.

Proposition 29. For a smooth manifold M, $\xi \in TM$ and a real number $a > 0$, $I_{a\xi} = \frac{1}{a} I_\xi$ and $\gamma_{a\xi}(t) = \gamma_\xi(at)$ where γ is the unique maximal geodesic.

Proof. Omit. It is from the uniqueness of the maximal geodesic.

1.6 Exponential map

Proposition 30. A set $\mathcal{T}M = \{\xi \in TM \mid 1 \in I_\xi\}$ is an open starlike with respect to $0 \in \Gamma(TM)$, which means 0 vector field.

Proof. Open is from the openness of maximal interval of the maximal flow and starlike is from the fact that $\xi \in \mathcal{T}M$ implies $a\xi \in \mathcal{T}M$ for every $a \in [0, 1]$.

Definition 31. For a smooth manifold M, the **exponential map** is a map $\exp : \mathcal{T}M \to M$ defined as $\exp \xi = \gamma_\xi(1)$.

Proposition 32. Exponential map is a differential map and has the maximal rank at $0 \in \Gamma(TM)$, which means for every reduced map $\exp_p : T_pM \cap \mathcal{T}M \to M$ defined as $\exp_p \xi = \exp \xi$ has the maximal rank at 0, so locally diffeomorphic at 0.

Proof. Differentiability is from the differentiability of flows. Now, fix $p \in M$ and let canonical identification be $I_0 : T_pM \to T_0(T_pM)$. Then, for any $\xi \in T_pM$, let $\omega_\xi : I_\xi \to T_pM$ as $\omega_\xi(t) = t\xi$. Then, by definition, $\omega'_\xi(0) = I_0\xi$. Now, $d(\exp_p)_0$ is a map from $T_0(\mathcal{T}M \cap T_pM)$ to T_pM since $\exp_p(0) = p$. Now, if $t \in I_\xi$ then $1 \in I_{t\xi}$, so $\exp_p \circ \omega_\xi(t) = \exp(t\xi) = \gamma_{t\xi}(1) = \gamma_\xi(t)$. Thus,

$$\begin{aligned} (d(\exp_p)_0 \circ I_0)(\xi) &= d(\exp_p)_0(\omega'_\xi(0)) = (d(\exp_p)_0 \circ d(\omega_\xi)_0)\frac{d}{dt} = d(\exp_p \circ \omega_\xi)_0 \frac{d}{dt} \\ &= (\exp_p \circ \omega_\xi)'(0) = \gamma'_\xi(0) = \xi \end{aligned}$$

which proves $d(\exp_p)_0 \circ I_0 = \mathrm{id}_{T_pM}$, so has maximal rank at 0. □

1.7 Torsion tensor and Curvature tensors

Definition 33. For a smooth manifold M with a connection, the **torsion tensor of vector fields** is a map $T : \Gamma^\infty(TM) \times \Gamma^\infty(TM) \to \Gamma^\infty(TM)$ which is defined as

$$T(X, Y) = \nabla_Y X - \nabla_X Y - [Y, X]$$

Proposition 34. The torsion tensor T has following properties.

1. $T(X, Y) + T(Y, X) = 0$.
2. $T(X + Z, Y) = T(X, Y) + T(Z, Y)$
3. $T(fX, Y) = fT(X, Y)$

Thus, if $X = \sum_j \xi^j \partial_j$ and $Y = \sum_j \eta^j \partial_j$, then $T(X, Y) = \sum_{j,k} \xi^j \eta^k T(\partial_j, \partial_k)$.

Proof. First, $T(X,Y)+T(Y,X)=0$ since $[X,Y]+[Y,X]=0$. $T(X+Z,Y)=T(X,Y)+T(Z,Y)$ is from the multilinearity of the ∇ and Lie derivatives. Lastly, for $f\in C^\infty(M)$,

$$\begin{aligned}T(fX,Y)&=\nabla_Y(fX)-\nabla_{fX}Y-[Y,fX]=(Yf)X+f\nabla_YX-f\nabla_XY-(Y(fX)-fXY)\\&=(Yf)X+f(\nabla_YX-\nabla_XY)-(Yf)X-fYX+fXY\\&=f(\nabla_YX-\nabla_XY-[Y,X])=fT(X,Y)\end{aligned}$$

□

Remark. Using this fact, if we fix $p\in M$ and choose $\xi,\eta\in T_pM$, then for any vector field X,Y such that $X|_p=\xi, Y|_p=\eta$, $T(X,Y)|_p$ is independent from the choice of X,Y. Thus, we can define the **torsion tensor** $T:T_pM\times T_pM\to T_pM$ as $T(\xi,\eta)=\nabla_\eta X-\nabla_\xi Y-[Y,X]|_p$, where X,Y are extensions of ξ,η, respectively, defined in a neighborhood of p.

Definition 35. For a smooth manifold M with a connection, **curvature tensor** is a map $R:T_pM\times T_pM\times T_pM\to T_pM$ is defined as

$$R(\xi,\eta,\zeta)=R(\xi,\eta)\zeta=\nabla_\eta\nabla_XZ-\nabla_\xi\nabla_YZ-\nabla_{[Y,X]|_p}Z$$

where X,Y,Z are extensions of ξ,η,ζ respectively, defined in a neighborhood of p, and above definition is well-defined but we will omit the proof of it.

Remark. Similar to the case of T, R is multilinear over functions, and $R(X,Y)Z+R(Y,X)Z=0$.

Proposition 36. If $T\equiv 0$ on whole manifold M, then for every $\xi,\eta,\zeta\in T_pM$, $R(\xi,\eta)\zeta+R(\zeta,\xi)\eta+R(\eta,\zeta)\xi=0$. This is called as **first Bianchi identity**.

Proof. Since multilinear, it is enought to show when $\xi=\partial_j|_p,\eta=\partial_k|_p,\zeta=\partial_l|_p$. Choose extensions as $\partial_j,\partial_k,\partial_l$. Since $T\equiv 0$, $T(\partial_j,\partial_k)=\nabla_{\partial_k}\partial_j-\nabla_{\partial_j}\partial_k-[\partial_k,\partial_j]=0$. Since $[\partial_k,\partial_j]=0$, $\nabla_{\partial_k}\partial_j=\nabla_{\partial_j}\partial_k$. Thus, $R(\partial_j|_p,\partial_k|_p)\partial_l|_p=\nabla_{\partial_k|_p}\nabla_{\partial_j}\partial_l-\nabla_{\partial_j|_p}\nabla_{\partial_k}\partial_l$. Then, use $\nabla_{\partial_k}\partial_j=\nabla_{\partial_j}\partial_k$, it is easily shown that $R(\partial_j,\partial_k)\partial_l+R(\partial_l,\partial_j)\partial_k+R(\partial_k,\partial_l)\partial_j=0$. □

Definition 37. For a smooth manifold M with a connection, T^l_{jk} and R^l_{ijk} in a hidden chart are defined as functions satisfy followings

1. $T(\partial_j,\partial_k)=\sum_l T^l_{jk}\partial_l$
2. $R(\partial_j,\partial_k)\partial_i=\sum_l R^l_{ijk}\partial_l$

Remark. By just calculation, $T^l_{jk}=\Gamma^l_{jk}-\Gamma^l_{kj}$. Similarily, R^l_{ijk} is also a function of Christoffel symbols, $R^l_{ijk}=\partial_k\Gamma^l_{ij}-\partial_j\Gamma^l_{ik}+\sum_r(\Gamma^r_{ij}\Gamma^l_{rk}-\Gamma^r_{ik}\Gamma^l_{rj})$

1.8 Riemannian metric

Definition 38. For a smooth manifold M, g is a **Riemannian metric** if $g:\bigcup_p\{p\}\times T_pM\times T_pM\to\mathbb{R}$ such that $g_p:T_pM\times T_pM\to\mathbb{R}$ is an inner product for every $p\in M$.

Definition 39. A Riemannian metric g is **differentiable** if for any open $U\subseteq M$ and differentiable vector fields X,Y over U, $g(X,Y)|_U:U\to\mathbb{R}$ is differentiable. Manifold is a **Riemannian manifold** if it equipped with a smooth Riemannian metric.

Definition 40. For smooth manifolds M,N with map $\phi:M\to N$ and a Riemannian metric h on N, **pull-back metric on** $M_0=\{p\in M\mid d\phi_p \text{ is injective}\}$ is defined as

$$\phi^*h_p(\xi,\eta)=h_{\phi(p)}(d\phi_p\xi,d\phi_p\eta)$$

Remark. If a point p is not in M_0, ϕ^*h_p can be defined as a symmetric bilinear map, but it would be not positive definite.

Definition 41. For smooth manifolds M,N with Riemannian metrics g,h respectively, $\phi:M\to N$ is a **local isometry of** M_0 **into** N if $g|_{M_0}=\phi^*h$. $\phi:M\to N$ is an **isometry of** M **into** N if ϕ is an embedding local isometry. $\psi:M\to M$ is an **isometry of** M if ψ is a diffeomorphic isometry.

For a manifold with a metric g, $g_p(\xi,\eta)$ is also denoted as $\langle\xi,\eta\rangle|_p$, or $\langle\xi,\eta\rangle_p$.

Definition 42. For a Riemannian manifold (M,g), ∇ is a **Levi-Civita Connection** if it is a connection satisfying

1. $\nabla_X Y = \nabla_Y X + [X, Y]$, which means $T \equiv 0$, torsion-free.
2. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

Remark. If Y, Z are vector fields along a curve ω, then $\frac{d}{dt}\langle Y, Z\rangle_{\omega(t)} = \langle \nabla_t Y, Z\rangle_{\omega(t)} + \langle Y, \nabla_t Z\rangle_{\omega(t)}$ is also satisfied.

Proposition 43. For a Riemannian manifold (M, g), there exists a unique Levi-Civita Connection.

Proof. If ∇ is a Levi-Civita Connection,

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle - \langle Y, \nabla_X Z\rangle = X\langle Y, Z\rangle - \langle Y, \nabla_Z X\rangle - \langle Y, [X, Z]\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle + \langle \nabla_Z Y, X\rangle - \langle Y, [X, Z]\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle + \langle \nabla_Y Z, X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle \\ &= X\langle Y, Z\rangle - Z\langle Y, X\rangle + Y\langle Z, X\rangle - \langle Z, \nabla_X Y\rangle - \langle Z, [Y, X]\rangle + \langle [Z, Y], X\rangle - \langle Y, [X, Z]\rangle\end{aligned}$$

which proves

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle - Z\langle Y, X\rangle + Y\langle Z, X\rangle - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle)$$

Thus, if Z_j are orthogonal basis, then $\nabla_X Y = \sum_j \frac{\langle \nabla_X Y, Z_j\rangle}{\langle Z_j, Z_j\rangle} Z_j$,, which proves $\nabla_X Y$ must be unique if exists. To prove existence, define

$$\nabla_X Y = \sum_j \frac{X\langle Y, Z_j\rangle - Z_j\langle Y, X\rangle + Y\langle Z_j, X\rangle - \langle X, [Y, Z_j]\rangle - \langle Y, [X, Z_j]\rangle - \langle Z_j, [Y, X]\rangle}{2\langle Z_j, Z_j\rangle} Z_j$$

Then, it can be done that ∇ is really a connection satisfying conditions of the Levi-Civita connection. Detail is omitted. □

Definition 44. For a Riemannian manifold (M, g) with a given chart, $g_{ij}(p) = g_p(\partial_i|_p, \partial_j|_p)$, $G(p)$ is the matrix consist by $g_{ij}(p)$, which is symmetric positive definite, and g^{ij} are components of $G^{-1}(p)$.

First condition of Levi-Civita connection gives $\Gamma^l_{jk} = \Gamma^l_{kj}$. Moreover, the formula of $\langle \nabla_X Y, Z\rangle$ gives

$$\langle \nabla_{\partial_k} \partial_j, \partial_r\rangle = \frac{1}{2}(\partial_k g_{jr} - \partial_r g_{jk} + \partial_j g_{rk})$$

Thus, by the definition of Christoffel symbols,

$$\sum_m \Gamma^m_{jk} g_{mr} = \frac{1}{2}(\partial_k g_{jr} - \partial_r g_{jk} + \partial_j g_{rk})$$

Then, use

$$\sum_r \sum_m \Gamma^m_{jk} g_{mr} g^{rl} = \sum_m \Gamma^m_{jk} \delta_{ml} = \Gamma^l_{jk}$$

we can conclude

$$\Gamma^l_{jk} = \frac{1}{2} \sum_r g^{rl}(\partial_k g_{jr} - \partial_r g_{jk} + \partial_j g_{rk})$$

which is the formula of the Christoffel symbols of the Levi-Civita connection induced by the Riemannian metric.

Lastly, for formal calculations, we might use $ds^2 = \sum_{j,k} g_{jk} dx^j dx^k$ which is the classical expression for the Riemannian metric, which becomes $ds^2 = \sum_j (dx^j)^2$ for the usual Euclidean metric.

1.9 Lengths of Curves on a Riemannian manifold

Definition 45. For $\xi \in T_p M$, $|\xi| = \sqrt{\langle \xi, \xi\rangle_p}$.

Definition 46. For any continuous piecewise C^1 path $\omega : [a, b] \to M$, the **length of** ω is defined as

$$\ell(\omega) = \int_a^b |\omega'(t)| dt = \int_a^b \sqrt{\langle \omega'(t), \omega'(t)\rangle_{\omega(t)}}$$

Definition 47. For a connected Riemannian manifold M, so is path-connected, $d : M \times M \to \mathbb{R}$ is a function defined as

$$d(p, q) = \inf_{\omega \in \Omega} \ell(\omega)$$

where Ω is a space of continuous piecewise C^1 path start at p and end at q, which is well-defiend since $\ell(\omega) \geq 0$ always.

Proposition 48. For a connected Riemannian manifold M, d is indeed a metric of M.

Proof. From its definition, $d(p,q) = d(q,p) \geq 0$ and $d(p,q) \leq d(p,r) + d(r,q)$ are easily shown. Now, for $p \in M$, choose $x : U \to \mathbb{R}^n$ which is a chart of M satisfying $p \in U$. Then, choose $r > 0$ such that $\overline{B(x(p),r)} \subseteq x(U)$ which is always possible in the Euclidean space. Now, for each $q \in U$, define $\lambda(q) = \min_{|\xi|=1, \xi \in T_qM} \frac{|\xi|}{\sqrt{\sum_j (\xi^j)^2}}$ where $\xi = \sum_j \xi^j \partial_j$. Then, $\lambda > 0$. Moreover, for every $\xi \in T_qM$, $|\xi| \geq \lambda(q)\sqrt{\sum_j (\xi^j)^2}$. Then, choose $\lambda = \min_{q \in x^{-1}(\overline{B(x(p),r)})} \lambda(q)$ which exists from the compactness, and $\lambda > 0$. Then, for any $\xi \in T_qM$ with $q \in x^{-1}(B(x(p),r))$, $|\xi| \geq \lambda\sqrt{\sum_j (\xi^j)^2}$ which proves $d(p,q) \geq \lambda|x(p) - x(q)|$ for $q \in x^{-1}(B(x(p),r))$ where $|x(p) - x(q)|$ is computed by the Euclidean metric. Thus, especially, $p \neq q$ and $q \in x^{-1}(B(x(p),r))$ implies $d(p,q) > 0$. If $q \notin x^{-1}(B(x(p),r))$, then for any path $\omega : [a,b] \to M$ such that $\omega(a) = p, \omega(b) = q$, there exists $s \in [a,b]$ such that $x(\omega(s)) \in \partial B(x(p),r)$, which is from the continuity of the metric of the Euclidean space. Now, $\ell(\omega) \geq \ell(\omega|_{[a,s]}) \geq d(p,\omega(s)) \geq \lambda|x(p) - x(\omega(s))| = \lambda r$. Thus, $d(p,q) \geq \lambda r > 0$ which proves $d(p,q) > 0$ if $p \neq q$, so d is a metric of M. $\square$

There is another proof using geodesics, which actually proves stronger fact that in local sense, geodesic curves give the minimum length. We will do this from now.

Definition 49. For a Riemannian manifold M and $q \in$M, define $\mathsf{B}(q,\varepsilon) = \{\xi \in T_qM \mid |\xi| < \varepsilon\}$, $\mathsf{B}_q = \mathsf{B}(q,1)$, $\mathsf{S}(q,\varepsilon) = \{\xi \in T_qM \mid |\xi| = \varepsilon\}$, $\mathsf{S}_q = \mathsf{S}(q,1)$. Also, for $V \subseteq M$, define $\mathsf{B}(V,\varepsilon) = \bigcup_{q \in V} \mathsf{B}(q,\varepsilon)$ and $\mathsf{S}(V,\varepsilon) = \bigcup_{q \in V} \mathsf{S}(q,\varepsilon)$

Theorem 50. For a Riemannian manifold M and each point $p \in M$, there exist an $\varepsilon > 0$ and an open neighborhood $U \subseteq M$ such that

1. $\forall \alpha, \beta \in U$ are jointed by a unique geodesic curve γ such that $\ell(\gamma) < \varepsilon$.
2. The geodesic depends differentiably on the endpoints.
3. For every $q \in U$, $\exp_q$ maps $\mathsf{B}(q,\varepsilon)$ diffeomorphically onto an open set of M.

Proof. Omit detail. For $o : M \to TM$ such that $o(p) = 0_p \in T_pM$, use a kind of application of **Proposition 32.**, $\pi \times \exp : \mathcal{T}M \to M \times M$ also has the maximal rank on $o(M)$. Then, use inverse function theorem so local diffeomorphic property is obtained. Then let $W \subseteq \mathcal{T}M$ be an open set gives diffeomorphism. Then, choose V and $\varepsilon > 0$ satisfying $\mathsf{B}(V,\varepsilon) \subseteq W$ and choose U such that $U \times U \subseteq (\pi \times \exp)(\mathsf{B}(V,\varepsilon))$.

Proposition 51. For a Riemannian manifold M and geodesic γ_ξ, $|\gamma_\xi'(t)| = |\xi|$.

Proof. Easily, $\frac{d}{dt}\langle \gamma_\xi'(t), \gamma_\xi'(t)\rangle = 2\langle \nabla_t \gamma_\xi'(t), \gamma_\xi'(t)\rangle = 2\langle 0, \gamma_\xi'(t)\rangle = 0$ by the definition of geodesic curves. Thus, $\frac{d}{dt}|\gamma_\xi'(t)|^2 = 0$ so $|\gamma_\xi'(t)|$ is constant along $\gamma_\xi(t)$. Since $\gamma_\xi'(0) = \xi$, $|\gamma_\xi'(t)| = |\gamma_\xi'(0)| = |\xi|$

Lemma 52 (Gauss Lemma)**.** Let $p \in M$ and $\mathsf{B}(p,\delta_0) \subseteq \mathcal{T}M$. Then, for any $t \in (0,\delta_0), \xi \in \mathsf{S}_p, \zeta \in T_{t\xi}\mathsf{S}(p,t)$, $\langle \gamma_\xi'(t), d(\exp_p)_{t\xi}\zeta\rangle = 0$.

Proof. Fix $\xi \in \mathsf{S}_p$. We may assume $\dim M \geq 2$. Define $\xi^\perp = \{\eta \in T_pM \mid \langle \eta, \xi\rangle = 0\}$. Let $I_{t\xi} : T_pM \to T_{t\xi}T_pM$ be the canonical isomorphism. Now, fix $\eta \in \xi^\perp$ and $t \in (0,\delta_0)$, define a path in T_pM as $\omega_t(\theta) = t(\cos(|\eta|\theta)\xi + \sin(|\eta|\theta)\frac{\eta}{|\eta|})$. Then, $\omega_t(0) = t\xi$ and $\omega_t'(0) = tI_{t\xi}\eta$. Now, $\omega_t(\theta) \in \mathsf{S}(p,t)$, so $tI_{t\xi}\eta \in T_{t\xi}\mathsf{S}(p,t)$. Because of the dimension, map $\eta \mapsto tI_{t\xi}\eta$ is actually an isomorphism. Then, for any $\zeta \in T_{t\xi}\mathsf{S}(p,t)$, $\eta = t^{-1}I_{t\xi}^{-1}\zeta \in \xi^\perp$. Let $v(t,\theta) = \exp \omega_t(\theta)$ which is well-defined since $|\omega_t(\theta)| = t < \delta_0$. Then, $v(t,0) = \exp \omega_t(0) = \exp t\xi = \gamma_{t\xi}(1) = \gamma_\xi(t)$. Let $c(t) = (t,0)$ and $d_t(\theta) = (t,\theta)$. Denote $dv_{(t,0)}\partial_t$ as $\partial_t v$ and $dv_{(t,0)}\partial_\theta = \partial_\theta v$. Since $\partial_t|_{(t,0)} = c'(t)$ and $\partial_\theta|_{(t,0)} = d_t'(\theta)$, $\partial_t v(t,0) = dv_{(t,0)}c'(t) = (v \circ c)'(t) = \gamma_\xi'(t)$ and $\partial_\theta v(t,0) = dv_{(t,0)}d_t'(0) = (v \circ d_t)'(0) = \frac{\partial}{\partial\theta}\exp \omega_t(\theta)(0) = d(\exp_p)_{t\xi}\omega_t'(0) = d(\exp_p)_{t\xi}\zeta$. Thus, it is enought to show that $\langle \partial_t v, \partial_\theta v\rangle_{v(t,0)} = 0$.

Now, consider $\nabla_t = \nabla_{\partial_t}$ and $\nabla_\epsilon = \nabla_{\partial_\epsilon}$, where they are derivative along v. By **Proposition 51.**, $|\partial_t v| = 1$ all over (t,θ), and by definition of the geodesic curve, $\nabla_t \partial_t v = 0$. Moreover, using $\nabla_{\partial_j}\partial_k = \nabla_{\partial_k}\partial_j$, it can be done that $\nabla_t \partial_\theta v = \nabla_\theta \partial_t v$ generally. Then, we get

$$\begin{aligned}\frac{d}{dt}\langle \partial_t v, \partial_\theta v\rangle_{v(t,0)} &= \langle \nabla_t \partial_t v, \partial_\theta v\rangle_{v(t,0)} + \langle \partial_t v, \nabla_t \partial_\theta v\rangle_{v(t,0)} \\ &= \langle \partial_t v, \nabla_\theta \partial_t v\rangle_{v(t,0)} \\ &= \frac{1}{2}\frac{d}{d\theta}\langle \partial_t v, \partial_t v\rangle_{v(t,0)} = \frac{1}{2}\frac{d}{d\theta}1 = 0\end{aligned}$$

Lastly, $v(0,\theta) = \exp \omega_0(\theta) = \exp_p 0 = p$ which prove $\partial_\theta v|_{v(0,0)} = 0$. Thus, $\langle \partial_t v, \partial_\theta v\rangle_{v(t,0)} = \langle \partial_t v, \partial_\theta v\rangle_{v(0,0)} = 0$. $\square$

Lemma 53. For a Riemannian manifold M and $p \in M$, suppose U, ε satisfying **Theorem 50.**. Then, for any piecewise C^1 curve $\sigma : [\alpha, \beta] \to U \setminus \{p\}$ which can be written as the form $\sigma(\tau) = \exp_p t(\tau)\xi(\tau)$, where $t : [\alpha, \beta] \to (0, \varepsilon)$ and $\xi : [\alpha, \beta] \to \mathsf{S}_p$, $\ell(\sigma) \geq |t(\beta) - t(\alpha)|$. Moreover, equality hold if and only if t is monotone and ξ is constant.

Proof. Omit.

Theorem 54. Suppose $\gamma : [0, 1] \to M$ is a geodesic such that $\gamma(0) = p, \gamma(1) = q$ where $p, q \in U$, $\ell(\gamma) < \varepsilon$ where U, ε is from **Theorem 50.**. Then, for any continuous piecewise C^1 path $\omega : [0, 1] \to M$ satisfying $\omega(0) = p, \omega(1) = q$, $\ell(\gamma) \leq \ell(\omega)$. Moreover, equality holds only if $\gamma([0, 1]) = \omega([0, 1])$.

Proof. Choose $\varepsilon_0 > 0$ satisfying $(\varepsilon_0 + 1)\ell(\gamma) < \varepsilon$ and if we extend geodesic, so construct $\tilde{\gamma} : [-\varepsilon_0, 1] \to U \subseteq M$, ϵ, U satisfy **Theorem 50.** respect to $\tilde{\gamma}(-\varepsilon_0)$ also. Remark that for geodesics, $\beta : [t_1, t_2] \to M$, $\ell(\beta) = \int_{t_1}^{t_2} |\beta'(t)| dt = \int_{t_1}^{t_2} |\beta'(0)| dt = |\beta'(0)|(t_2 - t_1)$. Thus, $\ell(\tilde{\gamma}) = (\varepsilon_0 + 1)\ell(\gamma)$. Let $A = \mathsf{B}(\tilde{\gamma}(-\epsilon_0), \ell(\tilde{\gamma})) \setminus \overline{\mathsf{B}(\tilde{\gamma}(-\epsilon_0), \epsilon_0 \ell(\gamma))}$ which is an open annulus in $T_{\tilde{\gamma}(-\varepsilon_0)}M$ and $G = \exp_{\tilde{\gamma}(-\epsilon_0)} A$. Then, G is open in M, so $\{\tau \mid \omega(\tau) \in G\}$ is an open set in $\mathbb{R}$, so is a countable union of disjoint intervals (α_i, β_i). Then, for each interval, use **Lemma 53.**, $\ell(\omega|_{[\alpha_i, \beta_i]}) \geq |t_i(\beta_i) - t_i(\alpha_i)|$, where, by continuiuty, $\omega(\beta_i), \omega(\alpha_i)$ must on boundary of G. Thus, $t_i(\beta_i), t_i(\alpha_i) \in \{\ell(\tilde{\gamma}), \epsilon_0 \ell(\gamma)\}$. Thus, $|t_i(\beta_i) - t_i(\alpha_i)|$ is 0 or $\ell(\tilde{\gamma}) - \ell(\gamma)\epsilon_0 = \ell(\gamma)$. Hence,

$$\ell(\omega) \geq \sum_i \ell(\omega|_{[\alpha_i, \beta_i]}) = \sum_i |t_i(\beta_i) - t_i(\alpha_i)|$$

where at least one of $|t_i(\beta_i) - t_i(\alpha_i)|$ must be $\ell(\gamma)$. Thus, $\ell(\omega) \geq \ell(\gamma)$. Equality condition is also from **Lemma 53.**. ☐

This theorem implies a geodesic is the local minimizer of the length.

Remark. Moreover, it can be done easily that if a curve is unit speed piecewise C^2 and length minimizing, then it is a geodesic. It is from that, for length minimizing unit speed curve, any part of this curve is also length minimizing.

Corollary 55. If $p \neq q$ then $d(p, q) > 0$. Moreover, if $B(p, \delta) = \{q \in M \mid d(p, q) < \delta\}, S(p, \delta) = \{q \in M \mid d(p, q) = \delta\}$, ε is from **Theorem 50.**, then $\delta \in (0, \varepsilon)$ implies $B(p, \delta) = \exp \mathsf{B}(p, \delta)$, $S(p, \delta) = \exp \mathsf{S}(p, \delta)$, where exp is a diffeomorphism betwen them.

1.10 Hopf-Rinow's Theorem

Definition 56. A Riemannian manifold is **geodesically complete** if $\forall \xi \in TM$, $I_\xi = \mathbb{R}$ where I_ξ is the maximal interval where geodesic can be defined. In other words, exp is defined on all of TM.

Theorem 57 (Hopf-Rinow 1)**.** If a Riemannian manifold M is connected and geodesically complete, then every two points in M has a length minimizing joining geodesic curve.

Proof. Suppose $p, q \in M$ and $d(p, q) = \delta > 0$. If $\delta < \varepsilon$ where ε is from **Theorem 50.**, then done. Thus, we may assume $\delta \geq \varepsilon$. Now, choose $\delta_0 \in (0, \varepsilon)$. Then, by compactness, there exists $p_0 \in S(p, \delta_0)$ such that $d(p_0, q) = d(S(p, \delta_0), q)$. Now, let $\xi = \frac{1}{\delta_0} \exp_p^{-1}|_{\mathsf{B}(p,\varepsilon)}(p_0)$, so $|\xi| \in \mathsf{S}_p$, $\exp \delta_0 \xi = p_0$. First, $\delta = d(p, q) \leq d(p, p_0) + d(p_0, q) = \delta_0 + d(p_0, q)$, so $d(p_0, q) \geq \delta - \delta_0$. Moreover, for any continuous piecewise path $\omega : [0, 1] \to M$ with $\omega(0) = p, \omega(1) = q$ choose $\alpha \in (0, 1)$ such that $\omega(\alpha) \in S(p, \delta_0)$. Then, $\ell(\omega) = \ell(\omega|_{[0,\alpha]}) + \ell(\omega|_{[\alpha,1]}) \geq \delta_0 + d(\omega(\alpha), q) \geq \delta_0 + d(p_0, q) \geq \delta$. Then, since $\inf_\omega \ell(\omega) = \delta$, so $\delta_0 + d(p_0, q) = \delta$. Thus, $d(q, \gamma_\xi(\delta_0)) = \delta - \delta_0$.

Now, assume $\delta_1 = \max\{t \in [\delta_0, \delta] \mid d(q, \gamma_\xi(t)) = \delta - t\} < \delta$. Let $p_1 = \gamma_\xi(\delta_1)$. $d(p, p_1) + \delta - \delta_1 = d(p, p_1) + d(p_1, q) \geq d(p, q) = \delta$ gives $d(p, p_1) \geq \delta_1$. Moreover, since p, p_1 are joined by $\gamma_\xi|_{[0,\delta_1]}$ with $|\xi| = 1$, so $d(p, p_1) \leq \delta_1$ which proves $d(p, p_1) = \delta_1$. Then, consider ε_1 is from **Theorem 50.** where base point is p_1. Choose $\delta_2 > 0$ as $\delta_2 < \min\{\varepsilon_1, \delta - \delta_1\}$. Now, select $p_2 \in S(p_1, \delta_2)$ as $d(q, p_2) = d(q, S(p_1, \delta_2))$. Use similar argument, $\delta_2 + d(p_2, q) = d(p_1, q) = d(p, \gamma_\xi(\delta_1)) = \delta - \delta_1$. Thus,

$$\begin{aligned} d(p, q) = \delta &= \delta_1 + \delta_2 + d(p_2, q) \\ &= d(p, p_1) + d(p_1, p_2) + d(p_2, q) \\ &\geq d(p, p_2) + d(p_2, q) \\ &\geq d(p, q) \end{aligned}$$

so $d(p, p_2) = d(p, p_1) + d(p_1, p_2)$. By uniqueness of the geodesic and local minimizing property around p_1, $\gamma_\xi(\delta_1 + \delta_2) = p_2$. Thus, $d(q, \gamma_\xi(\delta_1 + \delta_2)) = \delta - \delta_1 - \delta_2$ and $\delta_1 + \delta_2 > \delta_1$ which is contradiction. Thus, $\delta_1 = \delta$, so $d(\gamma_\xi(\delta), q) = \delta - \delta = 0$ which implies $\gamma_\xi(\delta) = q$. ☐

Corollary 58. If a Riemannian manifold M is geodesically complete, then every closed bounded subset is compact.

Proof. For closed bounded set E, let $\delta = \overline{\sup\{d(p,q) \mid q \in E\}}$. Then, $B(p,\delta) = \exp \mathsf{B}(p,\delta)$, $S(p,\delta) \subseteq \exp \mathsf{S}(p,\delta) \subseteq \overline{B(p,\delta)}$. Thus, $E \subseteq \overline{B(p,\delta)} = \exp \overline{\mathsf{B}(p,\delta)}$ which is compact. Thus, E is a closed subset of a compact set, so is compact. $\square$

Remark. It proves that a geodesically complete space is a complete metric space.

Theorem 59 (Hopf-Rinow 2)**.** If a Riemannian manifold M is a complete metric space, then it is geodesically complete.

Proof. Suppose M is complete but not geodesically complete. Then, $\exists \xi \in TM$ such that $I_\xi \neq \mathbb{R}$. Without loss of generality, assume $\inf I_\xi = \alpha > -\infty$. Now, choose a decreasing sequence such that $t_n > \alpha$ and $t_n \to \alpha$. Now, $d(\gamma_\xi(t_n), \gamma_\xi(t_m)) \le \ell(\gamma_\xi|_{[\min\{t_n,t_m\},\max\{t_n,t_m\}]}) = |\xi||t_n - t_m|$ which proves $\gamma(t_n)$ is a Cauchy sequence. Therefore, $\exists q \in M$ such that $\gamma(t_n) \to q$ since M is a complete metric space. Now, let U be a neighborhood of q such that $\overline{U}$ is compact. Then, $\{\zeta \in \pi^{-1}(\overline{U}) \mid |\zeta| = |\xi|\}$ is compact in TM. Since $|\gamma_\xi'(t_n)| = |\xi|$ always, so there exists a subsequence τ_k of t_n such that $\gamma_\xi'(\tau_k)$ converges in TM. Assume $\gamma_\xi'(\tau_k) \to \eta$. Then, let $\beta : \{\alpha + x \mid x \in I_\eta\} \to M$ as $\beta(t) = \gamma_\eta(t-\alpha)$. Then, β is a geodesic curve, $\lim_{k\to\infty} \beta(\tau_k) = \gamma_\eta(0) = q = \lim_{k\to\infty} \gamma_\xi(\tau_k)$, $\lim_{k\to\infty} \beta'(\tau_k) = \gamma_\eta'(0) = \eta = \lim_{k\to\infty} \gamma_\xi'(\tau_k)$. Thus, by uniqueness of the geodesic with C^2 property and local diffeomorphic property of exp, $\beta(t) = \gamma_\xi(t)$ where both are defined. Thus,

$$\tilde{\gamma}(t) = \begin{cases} \gamma_\xi(t) & t \in I_\xi \\ \gamma_\eta(t-\alpha) & t \in \alpha + I_\eta \end{cases}$$

is a well-defined, geodesic curve since it is an integral curve of the geodesic field. Moreover, $\tilde{\gamma}(0) = \gamma_\xi(0)$, $\tilde{\gamma}'(0) = \xi$, and infimum of domain is $\alpha + \inf I_\eta < \alpha$ which is contradiction to that I_ξ is the maximal interval that geodesic start with ξ is defined and $\inf I_\xi = \alpha$. $\square$

1.11 Calculation with Moving Frames

Definition 60. For a real vector space E and $\alpha, \beta \in E^*$, $\alpha \wedge \beta : E \times E \to \mathbb{R}$ is defined as $(\alpha \wedge \beta)(\xi, \eta) = \alpha(\xi)\beta(\eta) - \alpha(\eta)\beta(\xi)$.

Definition 61. For a differentiable 1-form ω on M, which means $\omega_p \in T_pM^*$ for every $p \in M$, $d\omega$ satisfying $d\omega_p : T_pM \times T_pM \to \mathbb{R}$ for every $p \in M$ is defined as $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$

Remark. If $\omega = \sum_j \omega_j dx^j$, then $d\omega = \sum_{j<k} (\frac{\partial \omega_k}{\partial x^j} - \frac{\partial \omega_j}{\partial x^k}) dx^j \wedge dx^k$

Definition 62. Covariant derivative of a 1-form over a Manifold with a connection is $\nabla_X \omega$ which is defined as $\nabla_X \omega(Y) = X(w(Y)) - w(\nabla_X Y)$

Proposition 63.

$$d\omega(X,Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) - \omega(T(X,Y))$$

Proof.

$$\begin{aligned} d\omega(X,Y) &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ &= (\nabla_X \omega)(Y) + \omega(\nabla_X Y) - (\nabla_Y \omega)(X) - \omega(\nabla_Y X) - \omega([X,Y]) \\ &= (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) - \omega(\nabla_Y X - \nabla_X Y - [Y,X]) \\ &= (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X) - \omega(T(X,Y)) \end{aligned}$$

Definition 64. For an open $U \subseteq M$, differentiable vector fields $e_1, \cdots, e_n$ on U is a **moving frame** if they are pointwise linearly independent. In this case, denote its dual coframe field as $\omega^1, \cdots, \omega^n$, so $\omega^i(e_j) = \delta_{ij}$.

Theorem 65. For a Riemannian manifold M and orthonormal moving frame $e_1, \cdots, e_n$

1. $d\omega^j = \sum_k \omega^k \wedge \omega_k^j$ where ω_k^j are connection 1-forms defined as $\omega_j^k(\xi) = \langle \nabla_\xi e_j, e_k \rangle$ and satisfies $\omega_j^k = -\omega_k^j$.
2. $d\omega_j^k = \sum_l \omega_j^l \wedge \omega_l^k - \Omega_j^k$ where $\Omega_j^k(X,Y) = \langle R(X,Y)e_j, e_k \rangle$ and $\Omega_j^k = -\Omega_k^j$.

Proof. First, we will compute more generally. Consider any connection.

Suppose ω_j^k are 1-forms satisfying $\nabla_\xi e_j = \sum_k \omega_j^k(\xi) e_k$ which exists since e_i are basis. Then, $\omega^l(e_j)$ is constant, so $\xi \omega^l(e_j) = 0$ always. Then,

$$0 = \xi(\omega^l(e_j)) = (\nabla_\xi \omega^l)(e_j) + \omega^l(\nabla_\xi e_j) = (\nabla_\xi \omega^l)(e_j) + \omega_j^l(\xi)$$

Then, $\nabla_\xi \omega^l = \sum_j (\nabla_\xi \omega^l)(e_j)\omega^j = -\sum_j \omega_j^l(\xi)\omega^j$. Now,

$$\begin{aligned}
d\omega^j(X,Y) - \sum_k (\omega^k \wedge \omega_k^j)(X,Y) &= X(\omega^j(Y)) - Y(\omega^j(X)) - \omega^j([X,Y]) - \sum_k (\omega^k(X)\omega_k^j(Y) - \omega^k(Y)\omega_k^j(X)) \\
&= X(\omega^j(Y)) - Y(\omega^j(X)) - \omega^j([X,Y]) + (\nabla_Y \omega^j)(X) - (\nabla_X \omega^j)(Y) \\
&= \omega^j(\nabla_X Y) - \omega^j(\nabla_Y X) - \omega^j([X,Y]) \\
&= -\omega^j(\nabla_Y X - \nabla_X Y - [Y,X]) = -\omega^j(T(X,Y))
\end{aligned}$$

Now, consider following

$$\begin{aligned}
\nabla_Y \nabla_X e_j &= \nabla_Y \sum_k \omega_j^k(X) e_k \\
&= \sum_k (Y(\omega_j^k(X))e_k + \omega_j^k(X)\nabla_Y e_k) \\
&= \sum_k (Y(\omega_j^k(X))e_k + \sum_l \omega_j^k(X)\omega_k^l(Y)e_l) \\
&= \sum_k (Y(\omega_j^k(X)) + \sum_l \omega_j^l(X)\omega_l^k(Y))e_k
\end{aligned}$$

This gives

$$\nabla_Y \nabla_X e_j - \nabla_X \nabla_Y e_j - \nabla_{[Y,X]} e_j = \sum_k (d\omega_j^k(Y,X) + \sum_l \omega_j^l \wedge \omega_l^k(X,Y))e_k$$

and hence,

$$R(X,Y)e_j = \sum_k (d\omega_j^k(Y,X) + \sum_l (\omega_j^l \wedge \omega_l^k)(X,Y))e_k$$

Thus, $\omega^k(R(X,Y)e_j) = d\omega_j^k(Y,X) + \sum_l (\omega_j^l \wedge \omega_l^k)(X,Y)$, so by defining $\Omega_j^k(X,Y) = \omega^k(R(X,Y)e_j)$, we gain $d\omega_j^k(X,Y) = \sum_l (\omega_j^l \wedge \omega_l^k)(X,Y) - \Omega_j^k(X,Y)$.

Finally, for a Riemannian manifold with the Levi-Civita connection, we gain $T = 0$, so $d\omega^j = \sum_k (\omega^k \wedge \omega_k^j)$. Also, $\nabla_\xi e_j = \sum_k \omega_j^k(\xi)e_k$ gives $\omega_j^k(\xi) = \langle \nabla_\xi e^j, e_k\rangle$. Since $\xi\langle e_j, e_k\rangle = 0$, we get $\langle \nabla_\xi e_j, e_k\rangle + \langle e_j, \nabla_\xi e_k\rangle$, which gives $\omega_j^k = -\omega_k^j$. $\Omega_j^k = -\Omega_k^j$ is from $d\omega_j^k = -d\omega_k^j$. □

Remark. This proves $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$ since $\Omega_j^k(X,Y) = \omega^k(R(X,Y)e_j) = \langle R(X,Y)e_j, e_k\rangle$.

Remark. If we choose $\xi = e_m$, then $\omega_j^k(e_m) = \langle \nabla_{e_m} e_j, e_k\rangle = \langle \sum_r \Gamma_{jm}^r e_r, e_k\rangle = \sum_r \Gamma_{jm}^r g_{rk}$.

2 Riemannian Curvature

2.1 Riemannian Sectional Curvature

Recall $R(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z$ satisfying $R(X,Y)Z = -R(Y,X)Z$ and $R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z$. Thus, if $\dim M = 1$, then $R(X,Y)Z = R(X,fX)Z = fR(X,X)Z = 0$, so we may assume $\dim M \geq 2$ in this section.

Also, if $T \equiv 0$, we gain $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$. Moreover, $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$ is also true if ∇ is the Levi-Civita connection.

Proposition 66. $k(\xi,\eta) = \langle R(\xi,\eta)\xi,\eta\rangle$ determines R. In particular,

$$\langle R(\xi,\eta)\zeta,\mu\rangle = \frac{1}{6}\frac{\partial^2}{\partial s\partial t}\bigg|_{s=t=0}(k(\xi+s\zeta,\eta+t\mu) - k(\xi+s\mu,\eta+t\zeta))$$

Proof. Omit. Just calculate directly.

Definition 67. R_1 is defined as $R_1(\xi,\eta)\zeta = \langle\xi,\zeta\rangle\eta - \langle\eta,\zeta\rangle\xi$ and $k_1(\xi,\eta) = \langle R_1(\xi,\eta)\xi,\eta\rangle$.

Proposition 68. R_1 satisfies

1. $R_1(X,Y)Z = -R_1(Y,X)Z$
2. $R_1(X,Y)Z + R_1(Z,X)Y + R_1(Y,Z)X = 0$.
3. $\langle R_1(X,Y)Z,W\rangle = \langle R_1(Z,W)X,Y\rangle$.
4. $\langle R_1(X,Y)Z,W\rangle = -\langle R_1(X,Y)W,Z\rangle$.

Moreover, $k_1(\xi,\eta) = \langle\xi,\xi\rangle\langle\eta,\eta\rangle - \langle\xi,\eta\rangle^2 = |\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2$.

Proof. Omit.

Definition 69. K is defined as $K(\xi,\eta) = \frac{k(\xi,\eta)}{k_1(\xi,\eta)}$ which is called as **sectional curvature of the 2-section determined by** ξ,η, where this definition works when ξ,η are linearly independent.

Proposition 70. K only depends on the 2-dimensional subspace determined by ξ and η.

Proof. By simple calculation, $k(\alpha\xi+\beta\eta,\gamma\xi+\delta\eta) = (\alpha\delta-\beta\gamma)^2k(\xi,\eta)$ and $k_1(\alpha\xi+\beta\eta,\gamma\xi+\delta\eta) = (\alpha\delta-\beta\gamma)^2k_1(\xi,\eta)$. □

Theorem 71. If $\dim M = 2$, K is called as the **Gauss curvature**, which satisfies $R(\xi,\eta)\zeta = K(p)R_1(\xi,\eta)\zeta$.

Proof. Suppose $\{e_1,e_2\}$ is an orthonormal basis of T_pM. It is enought to show only the case $\xi=\zeta=e_1,\eta=e_2$. Let $\mathsf{R}(\xi) = R(e_1,\xi)e_1$. Then, R is self-adjoint, so orthogonally diagonalizable. Moreover, e_1 is an eigenvector with eigenvalue 0, so e_2 is another eigenvector. Now, $\langle\mathsf{R}e_2,e_2\rangle = \langle R(e_1,e_2)e_1,e_2\rangle = k(e_1,e_2) = K(p)k_1(e_1,e_2) = K(p)$ is the eigenvalue of e_2 since e_1,e_2 are orthonormal. Thus, $R(e_1,e_2)e_1 = K(p)e_2 = K(p)R_1(e_1,e_2)e_1$, so is done. □

Definition 72. **Ricci curvature tensor** is a map $T_pM\times T_pM\to\mathbb{R}$ such that $Ric(\xi,\eta)$ is defined as the trace of the map $\zeta\mapsto R(\xi,\zeta)\eta$. In other words, for orthonormal basis e_i, $Ric(\xi,\eta) = \sum_j\langle R(\xi,e_j)\eta,e_j\rangle$.

Proposition 73. Ric is a symmetric bilinear form.

Proof. Omit.

Easily, if we choose $e_n = \frac{\xi}{|\xi|}$ to generate a orthonormal basis, then

$$K(\xi,\xi) = \sum_j\langle R(\xi,e_j)\xi,e_j\rangle = \sum_j K(\xi,e_j)|\xi|^2 = (\sum_{j=1}^{n-1}K(\xi,e_j))|\xi|^2$$

Then, to calculate associated quadratic form, scalar curvature S, will be defined as $S = \sum_{j,k=1,j\neq k}^n K(e_j,e_k)$.

For the Ricci curvature, if $Ric_{ik} = Ric(e_i,e_k)$, flow satisfying

$$\begin{cases}\frac{\partial g_{ij}(t)}{\partial t} = -2Ric_{ij}\\ g_{ij}(0) = g_{ij}^0\end{cases}$$

is called as the Ricci flow.

Definition 74. R_{ijkl} is defined as $R_{ijkl} = \langle R(e_i,e_j)e_k,e_l\rangle$.

2.2 Riemannian submanifold

In this section, consider $\varphi : M \to \overline{M}$, an isometric embedding of M into $\overline{M}$, as an inclusion.

Definition 75. If M is an embedded Riemannian submanifold of a Riemannian manifold $\overline{M}$, then $(T_pM)^{\perp}$ is the orthogonal complement of T_pM in $T_p\overline{M}$. For each $\overline{\xi} \in T_p\overline{M}$, $\overline{\xi}^T$ is the projection of $\overline{\xi}$ on T_pM which is called as the **tangential part** and $\overline{\xi}^N$ is the projection on $(T_pM)^{\perp}$ which is called as the **normal part**. The **normal bundle** of M in $T\overline{M}$ is defined as $\bigcup_{p\in M}(T_pM)^{\perp} = \nu M$. Then, $\Gamma(\nu M)$ is the set of differentiable sections of νM, which means a right inverse of π.

Proposition 76. For $p \in M$, $\xi \in T_pM$ and $Y \in \Gamma(TM)$, $\nabla_\xi Y = (\overline{\nabla}_\xi Y)^T$. Moreover, there is a symmetric bilinear form $B : T_pM \times T_pM \to (T_pM)^{\perp}$ such that for any $\xi, \eta \in T_pM$ and Y satisfying $Y|_p = \eta$, $B(\xi, \eta) = (\overline{\nabla}_\xi Y)^N$, which is called as the **2nd fundamental form**. If we define $b_v(\xi, \eta)$ for $v \in (T_pM)^{\perp}$ as $\langle B(\xi, \eta), v\rangle$, then bilinear self-adjoint linear transform A^v determined by $b_v(\xi, \eta) = \langle A^v\xi, \eta\rangle$ is actually $A^v\xi = -(\overline{\nabla}_\xi V)^T$, where V is a vector field in $\Gamma(\nu M)$ such that extension of $v \in (T_pM)^{\perp}$.

Proof. Let $D_\xi Y = (\overline{\nabla}_\xi Y)^T$ and $B(\xi, Y) = (\overline{\nabla}_\xi Y)^N$. At first, by calculation, $D_X Y$ can be shown as satisfying conditions of Levi-Civita connection, so is the Levi-Civita connection by uniqueness. Then, $B(X, Y) - B(Y, X) = (\overline{\nabla}_X Y - \overline{\nabla}_Y X)^N = ([X, Y])^N = 0$ since X, Y are tangential. Moreover, it gives well-definedness since $B(X, Y) = (\overline{\nabla}_\xi Y)^N = (\overline{\nabla}_\eta X)^N$, so only depends on $X|_p, Y|_p$. Finally, for $V \in \Gamma(\nu M)$,

$$\begin{aligned}
-\langle(\overline{\nabla}_\xi V)^T, \eta\rangle &= -\langle\overline{\nabla}_\xi V, \eta\rangle \\
&= -\xi\langle V, Y\rangle + \langle V, \overline{\nabla}_\xi Y\rangle \\
&= \langle V, \overline{\nabla}_\xi Y\rangle \\
&= \langle V, (\overline{\nabla}_\xi Y)^N\rangle = \langle v, B(\xi, \eta)\rangle
\end{aligned}$$

Theorem 77.

$$R(\xi, \eta)\zeta = (\overline{R}(\xi, \eta)\zeta)^T + A^{B(\xi,\zeta)}\eta - A^{B(\eta,\zeta)}\xi$$

Proof. Omit.

Remark. In other workds, $R(\xi, \eta)\zeta = (\overline{R}(\xi, \eta)\zeta)^T - (\overline{\nabla}_\eta B(\xi, \zeta))^N + (\overline{\nabla}_\xi B(\eta, \zeta))^N$. Thus,

$$\begin{aligned}
\langle R(\xi, \eta)\zeta, \mu\rangle &= \langle\overline{R}(\xi, \eta)\zeta, \mu\rangle + \langle-(\overline{\nabla}_\eta B(\xi, \zeta))^N, \mu\rangle - \langle-(\overline{\nabla}_\xi B(\eta, \zeta))^N, \mu\rangle \\
&= \langle\overline{R}(\xi, \eta)\zeta, \mu\rangle + \langle B(\eta, \mu), B(\xi, \zeta)\rangle - \langle B(\xi, \mu), B(\eta, \zeta)\rangle
\end{aligned}$$

This gives

$$K(\xi, \eta) = \overline{K}(\xi, \eta) + \frac{\langle B(\xi, \xi), B(\eta, \eta)\rangle - |B(\xi, \eta)|^2}{|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2}$$

2.3 Constant sectional curvature

Definition 78. A Riemannian manifold M has a **constant sectional curvature** κ if $K(\sigma) = \kappa$ for every 2-section σ.

Proposition 79. M has a constant sectional curvature κ if and only if $R(\xi, \eta)\zeta = \kappa(\langle\xi, \zeta\rangle\eta - \langle\eta, \zeta\rangle\xi)$.

Proof. Omit.

2.4 Second Fundamental Form Via Moving Frames

Assume $\dim(M) = n < m = \dim(\overline{M})$. Choose an orthonormal moving frame satisfy that $\overline{e}_1, \cdots, \overline{e}_n$ is an orthonormal moving frame of TM in $T\overline{M}$. Let $e_A = \overline{e}_A|_M$, $\omega^A = \overline{\omega}^A|_M$, $\omega^A_B = \overline{\omega}^A_B|_M$.

We already has $d\overline{\omega}^A = \sum_B \overline{\omega}^B \wedge \overline{\omega}^A_B$ where $\overline{\omega}^A_B = -\overline{\omega}^B_A$, $d\overline{\omega}^B_A = \sum_C \overline{\omega}^C_A \wedge \overline{\omega}^B_C - \overline{\Omega}^B_A$ where $\overline{\Omega}^B_A(X, Y) = \langle\overline{R}(X, Y)\overline{e}_A, \overline{e}_B\rangle$.

Then, for $j \le n$, $d\omega^j = \sum_k^n \omega^k \wedge \omega^j_k$. For $j > n$, $0 = d\omega^j = \sum_k^n \omega^k \wedge \omega^j_k$. Now, assume $\omega^\alpha_j = \sum_k h^\alpha_{jk}\omega^k$ where $\alpha > n$. Then, $0 = \sum_{j=1}^n \omega^j \wedge \sum_{k=1}^n h^\alpha_{jk}\omega^k = \sum j = 1^n \sum_{k=1}^n h^\alpha_{jk}\omega^j \wedge \omega^k$. Comparing coefficient, we get $h^\alpha_{jk} = h^\alpha_{kj}$. Lastly, $d\omega^k_j = \sum_{l=1}^n \omega^l_j \wedge \omega^k_l + \sum_{\alpha=n+1}^m \omega^\alpha_j \wedge \omega^k_\alpha - \overline{\Omega}^k_j|_M$. Thus, $\Omega^k_j = \overline{\Omega}^k_j|_M - \sum_{\alpha=n+1}^m \omega^\alpha_j \wedge \omega^k_\alpha$.

2.5 How the metric changes if we change coordinates.

Consider a metric is defined in $x_1, \cdots, x_n$ and $x_1, x_2, \cdots, x_n$ is a function of $y_1, \cdots, y_n$. Suppose g is the original metric and $\tilde{g}$ is the metric based on y_i. Now, $\frac{\partial}{\partial y_i} = \sum_j \frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}$. Then, $\tilde{g}_{ij} = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$. Thus,

$$\tilde{g}_{ij} = g(\sum_k \frac{\partial x_k}{\partial y_i}\frac{\partial}{\partial x_k}, \sum_l \frac{\partial x_l}{\partial y_j}\frac{\partial}{\partial x_l}) = \sum_k\sum_l \frac{\partial x_k}{\partial y_i} g_{kl} \frac{\partial x_l}{\partial y_j}$$

Suppose J is the Jacobian matrix $\frac{\partial(x_1,\cdots,x_n)}{\partial(y_1,\cdots,y_n)}$. Then $J_{ij}=\frac{\partial x_i}{\partial y_j}$. Then, above calculation gives

$$\tilde{g}_{ij}=\sum_k\sum_l J_{ki}g_{kl}J_{lj}=\sum_k\sum_l J^T_{ik}g_{kl}J_{lj}$$

which proves $\tilde{G}=J^TGJ$. In another way, denote metric as $\sum_{i,j}g_{ij}dx^idx^j$. Since $dx^i=\sum_j\frac{\partial x_i}{\partial y_j}dy^j$. Use this, metric becomes

$$\begin{aligned}\sum_{k,l}g_{kl}dx^kdx^l&=\sum_{k,l}g_{kl}(\sum_i\frac{\partial x_k}{\partial y_i}dy^i)(\sum_j\frac{\partial x_l}{\partial y_j}dy^j)\\&=\sum_{i,j,k,l}g_{kl}\frac{\partial x_k}{\partial y_i}\frac{\partial x_l}{\partial y_j}dy^idy^j\\&=\sum_{i,j}(\sum_{k,l}\frac{\partial x_k}{\partial y_i}g_{kl}\frac{\partial x_l}{\partial y_j})dy^idy^j\end{aligned}$$

which gives same result above. Actually, for above calculations, number of y_is and x_js not need to be same, where J will not be a square matrix. Lastly, for $x_1,\cdots,x_n$, function of $y_1,\cdots,y_m$, denote this map as ϕ. Then, $d\phi(\frac{\partial}{\partial y_i})f=\frac{\partial}{\partial y_i}(f\circ\phi)=\sum_j\frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial y_i}=\sum_j\frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}f$. Thus, $d\phi(\frac{\partial}{\partial y_i})=\sum_j\frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}$. Then, consider $\phi^*g(\frac{\partial}{\partial y_i},\frac{\partial}{\partial y_j})=g(\sum_k\frac{\partial x_k}{\partial y_i}\frac{\partial}{\partial x_k},\sum_l\frac{\partial x_l}{\partial y_j}\frac{\partial}{\partial x_l})$ which gives same results above, which means actually, above chain rule is just the pull-back metric.

2.6 Standard metrics of some sets

2.6.1 Unit sphere

Unit sphere $S^2\subseteq\mathbb{R}^3$, so S^2 has a natural Riemannian metric, induced by the Riemannian metric $dy_1^2+dy_2^2+dy_3^2$ of $\mathbb{R}^3$. Then, use the stereographic projection, $y_1=\frac{2x_1}{1+|x|^2},y_2=\frac{2x_2}{1+|x|^2},y_3=\frac{|x|^2-1}{1+|x|^2}$, it can be shown that $g(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})=g_{ij}=\frac{4\delta_{ij}}{(1+|x|^2)^2}$. From this, it is available to prove S^2 has the constant sectional curvature just from calculation.

Definition 80. Two metrics $g,\tilde{g}$ on M are **conformally equivalent** if $\exists f\in C^\infty(M)$ where $f>0$ such that $g(p)=f(p)\tilde{g}(p)$.

Remark. Since δ_{ij} is the metric $g_{\mathbb{R}^2}$, on $\mathbb{R}^2$, $g_{S^2\setminus\{N\}}$ and $g_{\mathbb{R}^2}$ are conformally equivalent. Moreover, use opposite stereographic projection, g_{S^2} is locally equivalent to $g_{\mathbb{R}^2}$. In other words, g_{S^2} is locally conforamally flat.

For hypersurface, there exist some diffrent models.

Definition 81. A map $g:\bigcup_p\{p\}\times T_pM\times T_pM\to\mathbb{R}$ is a **pseudometric** if it is symmetric, bilinear and $T_pM\ni\xi\neq0$ implies $\exists\eta\in T_pM$ such that $g(\xi,\eta)\neq0$.

For the Poincaré Ball model of hypersurface, we will give a metric to $B_1=\{x\in\mathbb{R}^n\mid\sum_j x_j^2<1\}$. For $n=2$ case, where it is nothing different for higher dimensions, let $U=\{(y_1,y_2,y_3)\in\mathbb{R}^3\mid y_1^2+y_2^2-y_3^2=-1,y_3>0\}$ equipped with the pseudometric $dy_1^2+dy_2^2-dy_3^2$. Then, use hyperbolic stereographic projection, $y_1=\frac{2x_1}{1-|x|^2},y_2=\frac{2x_2}{1-|x|^2},y_3=\frac{1+|x|^2}{1-|x|^2}$, which is from $\frac{(y_1,y_2)}{1+y_3}=(x_1,x_2)$. Then, this gives a metric on B_1 which is $\frac{4\delta_{ij}}{(1-|x|^2)^2}$.

Another model is the Halfspace model, that we define a metric on $(\mathbb{R}^n)^+=\{x\in\mathbb{R}^n\mid x_n>0\}$. Use diffeomorphism $\phi:B_1\to(\mathbb{R}^n)^+$ which is called as the Cayley transformation, a pull-back metric on B_1 of $\frac{\delta_{ij}}{x_n^2}$ on upper halfspace is $\frac{4\delta_{ij}}{(1-|x|^2)^2}$.

2.7 Compute the sectional curvature of S^n

Theorem 82. Suppose $v:(a,b)\times(-\varepsilon_0,\varepsilon_0)\to M$ is a differentiable map such that $\gamma_\varepsilon(t)=v(t,\varepsilon)$ is a geodesic curve for every $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$. Denote $dv_*\partial_t$ as ∂_tv and $dv_*\partial_\varepsilon$ as $\partial_\varepsilon v$. Moreover, denote ∇_{∂_t} as ∇_t and $\nabla_{\partial_\varepsilon}$ as ∇_ε where they are differentiation of vector fiels along v. Then, $\nabla_t\nabla_t\partial_\varepsilon v+\nabla_tT(\partial_tv,\partial_\varepsilon v)+R(\partial_tv,\partial_\varepsilon v)\partial_tv=0$. This is called as the **Jacobi Equation**.

Proof. Since $[\partial_t,\partial_\varepsilon]=0$, we get $\nabla_\varepsilon\partial_tv-\nabla_t\partial_\varepsilon v=T(\partial_tv,\partial_\varepsilon v)$ and $\nabla_\varepsilon\nabla_t-\nabla_t\nabla_\varepsilon=R(\partial_tv,\partial_\varepsilon v)$. Since γ_ε are geodesics, we get $\nabla_t\partial_tv=0$. Thus,

$$\begin{aligned}0&=\nabla_\varepsilon\nabla_t\partial_tv\\&=\nabla_t\nabla_\varepsilon\partial_tv+R(\partial_tv,\partial\varepsilon_v)\partial_tv\\&=\nabla_t(\nabla_t\partial_\varepsilon v+T(\partial_tv,\partial_\varepsilon v))+R(\partial_tv,\partial\varepsilon_v)\partial_tv\\&=\nabla_t\nabla_t\partial_\varepsilon v+\nabla_tT(\partial_tv,\partial_\varepsilon v)+R(\partial_tv,\partial_\varepsilon v)\partial_tv\end{aligned}$$

$\square$

Proposition 83. For a local isometry $\phi : M \to M$ for connected Riemannian manifold M, ϕ preserves

1. The distance
2. The Levi-Civita Connection
3. Geodesics
4. Sectional curvatures

Proof. The distance is preserved naturally by definition of the distance. To prove that the Levi-Civita connection is preserved, we may use Christoffel symbols, or prove that $\widetilde{\nabla}_X Y = d\phi(\nabla_{d\phi^{-1}X} d\phi^{-1}Y)$ satisfies the conditions of the Levi-Civita Connection and use uniqueness. Moreover, $d\partial_* \nabla_t V = \nabla_t(d\partial_* V)$ gives that geodesics are preserved. To prove this, choose V as elements of a basis, and use linearity. Sectional curvatures are also preserved easily. $\square$

Remark. Since geodesics are preserved, it gives that $\phi(\exp \xi) = \exp d_\phi \xi$

To compute the sectional curvature of $S^n(\rho)$, note that in $\mathbb{R}^k$, $\phi(p) = Ap + q$ is an isometry if A is an orthogonal transformation. Since $g_{S^n(\rho)}$ is the induced metric by inclusion, which is just a restricted metric of the canonical metric of the Euclidean space, $A|_{S^n(\rho)}$ for $A \in O(n+1)$ is an isometry of $S^n(\rho)$. Moreover, if we denote $I_p : \mathbb{R}^n \to T_p\mathbb{R}^n$ as the canonical identification, then $T_pS^n(\rho) = I_p(\{q \in \mathbb{R}^n \mid p \cdot q = 0\}) = I_p(p^\perp)$. Moreover, for $A \in O(n+1)$, $p \in S^n(\rho)$ and $\xi \in T_pS^n(\rho)$, $dA_p(\xi) = I_{Ap}(AI_p^{-1}(\xi))$.

Proposition 84. $S^n(\rho)$ has a constant sectional curvature.

Proof. For every $p, q \in S^n(\rho), \xi_1, \xi_2 \in T_pM, \eta_1, \eta_2 \in T_qM$ where $\langle \xi_1, \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle = 0$, since $\frac{p}{|p|}, I_p^{-1}(\xi_1), I_p^{-1}(\xi_2)$ and $\frac{q}{|q|}, I_q^{-1}(\eta_1), I_q^{-1}(\eta_2)$ are both orthonormal in $\mathbb{R}^{n+1}$, so there exists $A \in O(n+1)$ such that $q = Ap, \eta_1 = dA_p(\xi_1), \eta_2 = dA_p(\xi_2)$. Then, since A is an isometry, $K(\xi_1, \xi_2) = K(\eta_1, \eta_2)$ always, so $S^n(\rho)$ has a constant sectional curvature. $\square$

Proposition 85. Every geodesic of $S^n(\rho)$ is a part of the great circles

Proof. Choose $p, q \in S^n(\rho)$ such that there exists a unique geodesic $\gamma : [0, d(p,q)] \to S^n(\rho)$ such that $|\gamma'| = 1, \gamma(0) = p, \gamma(d(p,q)) = q$ which is always possible locally. Suppose σ is a plane passing through $0, p, q$, which is uniquely determined. Let A be the reflection to σ. Choose an orthonormal basis $\{e_1, \cdots, e_{n+1}\}$ of $\mathbb{R}^{n+1}$ as e_1, e_2 generates σ. Then, $Ae_1 = e_1, Ae_2 = e_2$ and $Ae_k = -e_k$ for $k \geq 3$. Thus, fixed points of A is just σ. Now, A is an isometry, so $A \circ \gamma$ is a geodesic of $S^n(\rho)$ and $|\xi| = |dA_p \xi|$ for every $\xi \in T_pM$. Moreover, $Ap = p, Aq = q$ implies $A \circ \gamma(0) = Ap = p = \gamma(0)$ and $A \circ \gamma(d[p,q]) = Aq = q = \gamma(d[0,q])$. Thus, $A \circ \gamma$ is also a length minimizing geodesic, so $A \circ \gamma = \gamma$. Thus, $\gamma([0, d(p,q)]) \subseteq \sigma \cap S^n(\rho)$ which is a great circle. Thus, every geodesic is a part of a great circle locally, so is a part of the great circles. $\square$

Theorem 86. $S^n(\rho)$ has a constant sectional curvature $\frac{1}{\rho^2}$.

Proof. First, fix $v_0 \in S^n(\rho)$. Choose $\xi, \eta \in T_{v_0}S^n(\rho)$ as orthonormal vectors. Define $v(t, \epsilon) = \exp_p t((\cos \epsilon)\xi + (\sin \epsilon)\eta)$. Note that $S^n(\rho)$ is complete, so geodesically complete, which means v is well-defined for every t, ϵ. Then, since geodesic is the great circle, $\gamma_\epsilon(t) = v(t, \epsilon)$ is the circle such that $\gamma_\epsilon(0) = v_0$ and $\gamma_\epsilon'(0) = (\cos \epsilon)\xi + (\sin \epsilon)\eta$. Thus, $\gamma_\epsilon(t) = \cos(ct)v_0 + \sin(ct)I_{v_0}^{-1}\rho((\cos \epsilon)\xi + (\sin \epsilon)\eta)$ for some constant c since $|v_0| = \rho$ in $\mathbb{R}^{n+1}$ and $|(\cos \epsilon)\xi + (\sin \epsilon)\eta| = 1$. $|\gamma_\epsilon'(0)| = 1$ gives $c = \frac{1}{\rho}$, so $\gamma_\epsilon(t) = (\cos(t/\rho))v_0 + (\sin(t/\rho))I_{v_0}^{-1}\rho((\cos \epsilon)\xi + (\sin \epsilon)\eta)$. Then, let $\gamma(t) = \gamma_0(t)$ and $Y(t) = (\partial_\epsilon v)(t, 0) = I_{\gamma(t)}(\sin(t/\rho))I_{v_0}^{-1}\rho\eta$. Then, by Jacobi-equation, $\nabla_t\nabla_t Y + R(\gamma', Y)\gamma' = 0$. Let $e(t) = I_{\gamma(t)}I_{v_0}^{-1}\eta$. Then, this is a parallel transport, so $\nabla_t e = 0$. Thus, $Y(t) = \rho(\sin(t/\rho))e(t)$ implies $\nabla_t\nabla_t Y(t) = -\rho\frac{1}{\rho^2}(\sin(t/\rho))e(t)$. Thus,

$$\begin{aligned} -\frac{1}{\rho}(\sin(t/\rho))e(t) &= \nabla_t\nabla_t Y \\ &= -R(\gamma'(t), Y(t))\gamma'(t) \\ &= -\rho(\sin(t/\rho))R(\gamma'(t), e(t))\gamma'(t) \end{aligned}$$

Then, for $t \neq 0$, we gain $\frac{1}{\rho}e(t) = \rho R(\gamma'(t), e(t))\gamma'(t)$. By continuity, it is also true for $t = 0$. Then, if $t = 0$, it gives $\frac{1}{\rho}\eta = \rho R(\xi, \eta)\xi$. Thus, $K(\xi, \eta) = \langle R(\xi, \eta)\xi, \eta \rangle = \frac{1}{\rho^2}\langle \eta, \eta \rangle = \frac{1}{\rho^2}$. $\square$

2.8 Variations of Arc Length

Definition 87. For a given $\omega : [a,b] \to M$ which is a continuous piecewise C^∞ path, $v : [a,b] \times (-\epsilon_0, \epsilon_0) \to M$ is a **variation of** ω if v is a continuous piecewise C^∞ such that $v(t,0) = \omega(t)$. v is a **homotopy of** ω if $v(a,\epsilon) = \omega(a), v(b,\epsilon) = \omega(b)$ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$, which means v fixes endpoints. v is a **smooth variation** if v is smooth over $[a,b] \times (-\epsilon_0, \epsilon_0)$.

Theorem 88. For differentiable $\omega : [a,b] \to M$ and a differentiable variation $v : [a,b] \times (-\epsilon_0, \epsilon_0) \to M$, let $L : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ as $L(\epsilon) = \ell(\omega_\epsilon)$ where $\omega_\epsilon(t) = v(t,\epsilon)$. In other words, $L(\epsilon) = \int_a^b |\partial_t v(t,\epsilon)| dt$. Then, L is differentiable and

$$\frac{dL}{d\epsilon} = \left\langle \partial_\epsilon v(t,\epsilon), \frac{\partial_t v(t,\epsilon)}{|\partial_t v(t,\epsilon)|} \right\rangle \Bigg|_{t=a}^{b} - \int_a^b \left\langle \partial_\epsilon v(t,\epsilon), \nabla_t \frac{\partial_t v(t,\epsilon)}{|\partial_t v(t,\epsilon)|} \right\rangle dt$$

Especially, if $|\omega'| = 1$ and $Y(t) = \partial_\epsilon v(t,0)$,

$$\frac{dL}{d\epsilon}(0) = \langle Y, \omega' \rangle|_a^b - \int_a^b \langle Y, \nabla_t \omega' \rangle dt$$

Proof.

$$\begin{aligned}
\frac{dL}{d\epsilon} &= \partial_\epsilon L = \partial_\epsilon \int_a^b \sqrt{\langle \partial_t v, \partial_t v \rangle} dt \\
&= \int_a^b \partial_\epsilon \sqrt{\langle \partial_t v, \partial_t v \rangle} dt \\
&= \int_a^b \frac{\partial_\epsilon \langle \partial_t v, \partial_t v \rangle}{2|\partial_t v|} dt \\
&= \int_a^b \frac{\langle \nabla_\epsilon \partial_t v, \partial_t v \rangle}{|\partial_t v|} dt \\
&= \int_a^b \frac{\langle \nabla_t \partial_\epsilon v, \partial_t v \rangle}{|\partial_t v|} dt \\
&= \int_a^b \left\langle \nabla_t \partial_\epsilon v, \frac{\partial_t v}{|\partial_t v|} \right\rangle dt \\
&= \int_a^b \partial_t \left\langle \partial_\epsilon v, \frac{\partial_t v}{|\partial_t v|} \right\rangle - \left\langle \partial_\epsilon v, \nabla_t \frac{\partial_t v}{|\partial_t v|} \right\rangle dt \\
&= \left\langle \partial_\epsilon v, \frac{\partial_t v}{|\partial_t v|} \right\rangle \Bigg|_{t=a}^{b} - \int_a^b \left\langle \partial_\epsilon v, \nabla_t \frac{\partial_t v}{|\partial_t v|} \right\rangle dt
\end{aligned}$$

□

Theorem 89. For a continuous piecewise C^∞ path $\omega : [a,b] \to M$ where $|\omega'| = 1$ whenever ω' is defined, ω is a geodesic if and only if $L'(0) = 0$ for every homotopy of ω.

Proof. By above theorem, if ω is a geodesic, $\nabla_t \omega' = 0$ gives $L'(0) = 0$ for every homotopy of ω.

Now, suppose $L'(0) = 0$ for every homotopy of the given ω. For every vector field Z along ω, define $\Delta Z(t_0) = \lim_{t \to t_0^+} Z(t) - \lim_{t \to t_0^-} Z(t)$. Then, for homotopy v of ω, we get $\frac{dL}{d\epsilon}(0) = -\sum_t \langle Y, \Delta\omega' \rangle - \int_a^b \langle Y, \nabla_t \omega' \rangle dt$ where summation only works for singular points which are finitely many, so is well-defined. Choose $t^* \in (a,b)$ which is not a singular point of ω'. Suppose $\nabla_t \omega'(t^*) \neq 0$. Choose $\delta > 0$ such that $(t^* - \delta, t^* + \delta)$ doesn't contain any singularity and $(t^* - \delta, t^* + \delta) \subseteq [a,b]$. Suppose Z is a parallel vector field on $(t^* - \delta, t^* + \delta)$ such that $Z(t^*) = \nabla_t \omega'(t^*)$. Then, by continuity, $\exists \delta_1 > 0$ such that $\langle Z, \nabla_t \omega' \rangle > 0$ for $|t - t^*| < \delta_1 \leq \delta$. Now let $\varphi : [a,b] \to [0,\infty)$ is a bump function such that $\varphi = 0$ for $t \notin (t^* - \delta_1, t^* + \delta_1)$, $\varphi > 0$ on at least an open interval. Define $Y = \varphi Z$ on $(t^* - \delta_1, t^* + \delta_1)$ and 0 otherwise. Finally, define $v(t,\epsilon) = \exp \epsilon Y(t)$. Then, $\partial_\epsilon v(t,0) = Y(t)$. Then, since $Y = 0$ whenever $\Delta\omega' \neq 0$, $L'(0) = 0$ gives $\int_a^b \langle Y, \nabla_t \omega' \rangle dt = \int_{t^*-\delta_1}^{t^*+\delta_1} \langle Y, \nabla_t \omega' \rangle dt = 0$ which is contradiction. Thus, $\nabla_t \omega' = 0$ where it is defined. Hence, for every homotopy v of ω, $L'(0) = -\sum_t \langle Y, \Delta\omega' \rangle$. Then, choose Y satisfying $Y(t) = \Delta\omega'(t)$ whenever $\Delta\omega'(t) \neq 0$, and define $v(t,\epsilon) = \exp \epsilon Y(t)$. It gives $0 = \sum_t \langle \Delta\omega'(t), \Delta\omega'(t) \rangle$ which proves ω is actually differentiable over $[a,b]$. Thus, ω is a geodesic. □

2.9 Jacobi Fields

Theorem 90. For a differentiable $\omega : [a,b] \to M$ with $|\omega'| = 1$, differentiable variation $v : [a,b] \times (-\epsilon_0, \epsilon_0) \to M$ and $L : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ satisfying $L(\epsilon) = \ell(v(-,\epsilon))$,

$$\frac{d^2 L}{d\epsilon^2}(0) = \langle \nabla_\epsilon \partial_\epsilon v, \omega' \rangle|_a^b + \int_a^b |\nabla_t Y|^2 - \langle R(\omega', Y)\omega', Y \rangle - \langle \omega', \nabla_t Y \rangle^2 - \langle \nabla_t \omega', \nabla_\epsilon \partial_\epsilon v \rangle dt$$

where $Y(t) = \partial_\epsilon v(t,0)$.

Proof. Omit.

Theorem 91. For a geodesic $\omega : [a,b] \to M$ with $|\omega'| = 1$, differentiable variation v, length function L and field $Y = \partial_\epsilon v|_{\epsilon=0}$,

$$L'(0) = \langle Y, \omega' \rangle|_a^b$$

and

$$L''(0) = \langle \nabla_\epsilon \partial_\epsilon v, \omega' \rangle|_a^b + \int_a^b |\nabla_t Y_\perp|^2 - \langle R(\omega', Y_\perp)\omega', Y_\perp \rangle dt$$

where $Y_\perp = Y - \langle Y, \omega' \rangle \omega'$

Proof. Use $(\nabla_t Y)_\perp = \nabla_t Y_\perp$ which is from $\nabla_t \omega' = 0$, it is just an application of **Theorem 88.** and **Theorem 90.**. □

Definition 92. For vector fields X, Y along a fixed geodesic $\gamma : [a,b] \to M$ with $|\gamma'| = 1$, $(X,Y) = \int_a^b \langle X, Y \rangle dt$.

Definition 93. For a geodesic $\gamma : [a,b] \to M$, γ_0 is the vector space of piecewise C^1 vector fields X along γ such that $X(a) = 0, X(b) = 0$ and orthogonal to γ.

Definition 94. Index form is a symmetric bilinear form defined on γ_0 defined as

$$I(X,Y) = \int_a^b \langle \nabla_t X, \nabla_t Y \rangle - \langle R(\gamma', X)\gamma', Y \rangle dt$$

Remark. If $Y \in \gamma_0$ is induced by some homotopy v of γ with the arc length function L, $L''(0) = I(Y,Y)$.

If we use the integration by parts, $I(X,Y) = -\int_a^b \langle \nabla_t^2 X + R(\gamma', X)\gamma', Y \rangle dt$.

Definition 95. Jacobi operator is a self adjoint operator associated to index form, which is defined as

$$\mathcal{L}X = -\nabla_t^2 X - R(\gamma', X)\gamma'$$

In other words, $(\mathcal{L}X, Y) = I(X,Y) = (X, \mathcal{L}Y)$.

Definition 96. Jacobi field is a vector field Y along a geodesic γ satisfying $\mathcal{L}Y = 0$, which is actually the Jacobi equation.

Remark. Easily, $\gamma', t\gamma'$ are Jacobi fields. Moreover, Jacobi equation is a linear equation.

Theorem 97. If $\mathcal{J}$ is the set of Jacobi fields, then $\mathcal{J}$ is a vector space with dimension $2\dim M$. Moreover, if geodesic is defined on $[a,b]$ and $t_0 \in [a,b]$ is fixed, then for every $\xi, \eta \in T_{\gamma(t_0)}M$, there is a unique $Y \in \mathcal{J}$ such that $Y(t_0) = \xi, (\nabla_t Y)(t_0) = \eta$. Furthermore, if $Y \in \mathcal{J}$ and $Y \neq 0$ at some point, then $|Y|^2 + |\nabla_t Y|^2 > 0$ on γ.

Proof. Suppose $Y(t) = \sum_j Y^j(t) E_j$ where E_j is a parallel transport and $E_j(t_0)$ are orthonormal. Then, $E_j(t)$ are also orthonormal. Then, define $\mathsf{R}_j^k = \langle R(\gamma', E_j)\gamma', E_k \rangle$. Now, $\mathcal{L}Y = 0$ if and only if $Y^{j''} + \sum_k Y^j \mathsf{R}_j^k = 0$ for every j. Thus, theorem follows from the ODE theory. □

Theorem 98. If $X, Y \in \mathcal{J}$, then $\langle \nabla_t X, Y \rangle - \langle X, \nabla_t Y \rangle$ is a constant. Moreover, for any $Y \in \mathcal{J}$, $\langle Y, \gamma' \rangle = at + b$ for some constant a, b. In other words, $\mathcal{J}^\perp = \{Y \in \mathcal{J} \mid \langle Y, \gamma' \rangle = 0\}$ is a subspace of $\mathcal{J}$ of codimension 2

Proof.

$$\partial_t(\langle \nabla_t X, Y \rangle - \langle X, \nabla_t Y \rangle) = \langle \nabla_t^2 X, Y \rangle - \langle X, \nabla_t^2 Y \rangle = \langle X, R(\gamma', Y)\gamma' \rangle - \langle Y, R(\gamma', X)\gamma' \rangle = 0.$$

Now,

$$\partial_t \langle Y, \gamma' \rangle = \langle \nabla_t Y, \gamma' \rangle + \langle Y, \nabla_t \gamma' \rangle = \langle \nabla_t Y, \gamma' \rangle - \langle Y, \nabla_t \gamma' \rangle$$

so is constant since $\gamma' \in \mathcal{J}$. Then, integrating both side gives result. □

2.10 Jacobi field of geodesic curves in Manifolds with constant sectional curvature

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If manifold has a constant sectional curvature κ, then for $Y \in \mathcal{J}^\perp$ on geodesic γ satisfying $|\gamma'| = 1$, $R(\gamma', Y)\gamma' = \kappa(\langle \gamma', \gamma' \rangle Y - \langle Y, \gamma' \rangle \gamma') = \kappa Y$ by **Proposition 79.**. Thus, Jacobi equation becomes $\nabla_t^2 Y + \kappa Y = 0$.

Definition 99. For κ, S_κ, C_κ are solutions of a differential equation $\psi'' + \kappa\psi = 0$ satisfying $S_\kappa(0) = 0, S'_\kappa(0) = 1, C_\kappa(0) = 1, C'_\kappa(0) = 0$. Thus,

$$C_\kappa(t) = \begin{cases} \cos\sqrt{\kappa}t & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh\sqrt{-\kappa}t & \kappa < 0 \end{cases}$$

$$S_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}t & \kappa > 0 \\ t & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}t & \kappa < 0 \end{cases}$$

Proposition 100. If manifold has a constant sectional curvature κ, then for given geodesic γ and $Y \in \mathcal{J}^\perp$, there exists a parallel vector fields A, B orthogonal to γ' such that $Y = C_\kappa A + S_\kappa B$.

Proof. It is just from the dimension argument. Also, if we choose a orthonormal vector field with $E_n = \gamma'$, then Jacobi equation gives ${Y^j}'' + \kappa Y^j = 0$ for $j = 1, 2, \cdots, \dim M - 1$ where $Y = \sum Y^j E_j$. Then, this result follows. ☐

2.11 Conjugate points

Definition 101. For a geodesic curve γ, $\gamma(t_1)$ is said to be **conjugate to** $\gamma(t_0)$ if there exists $Y \in \mathcal{J}$, $Y \not\equiv 0$ such that $Y(t_0) = Y(t_1) = 0$.

Remark. Naturally, such Y is in $\mathcal{J}^\perp$. Moreover, γ' has no zero point, and $t\gamma'$ has at most one zero point.

Proposition 102. For a geodesic curve $\gamma : [a, b] \to M$ and $t_0 \in (a, b]$ such that $\gamma(t_0)$ is not conjugate to $\gamma(a)$, for every $\xi \in \gamma'(t_0)^\perp$, there exists a unique $Y \in \mathcal{J}^\perp$ such that $Y(t_0) = \xi, Y(a) = 0$.

Proof. Omit.

Theorem 103. For a Riemannian manifold M with a constant sectional curvature κ and a unit speed geodesic $\gamma : (-\infty, \infty) \to M$, $\gamma(0)$ has a conjugate point if and only if $\kappa > 0$. In that case, $\gamma(\frac{l\pi}{\sqrt{\kappa}})$ are conjugate points.

Proof. Omit.

Theorem 104. Suppose $N \le \dim M - 1$ and $Y_1, \cdots, Y_N \in \mathcal{J}^\perp$ with $\langle Y_j, \nabla_t Y_k\rangle - \langle Y_k, \nabla_t Y_j\rangle = 0$ for every $1 \le j, k \le N$. Then, if $X = \sum f^j Y_j \in \gamma_0$, $I(X, X) = \int_a^b |\sum_j {f^j}' Y_j|^2 dt$.

Proof. Just from direct computation. We might get $|\nabla_t X|^2 - \langle R(\gamma', X)\gamma, X\rangle = |\sum {f^j}' Y_j|^2 + \langle \sum f^j \nabla_t Y_j, X\rangle'$. ☐

Theorem 105. If $\gamma : [a, b]$ has no conjugate point to $\gamma(a)$ over $(a, b]$, then I is a positive definite on γ_0. If no conjugate point over (a, b), then I is positive semidefinite and $I(X, X) = 0$ if and only if $X \in \mathcal{J}^\perp \cap \gamma_0$.

Proof. Choose linearly independent $Y_1, Y_2, \cdots, Y_{n-1} \in \mathcal{J}^\perp$ where $Y_1(a) = Y_2(a) = \cdots = Y_{n-1}(a) = 0$. Since no conjugate point on (a, b), there exists $f^j(t)$ such that $X(t) = \sum_{j=1}^{n-1} f^j(t) Y_j(t)$ on (a, b) since $X \in \gamma_0$. Now, if f^j are all bounded, then since $X(a) = X(b) = 0$, so

$$\begin{aligned} I(X, X) &= \int_a^b |\nabla_t X|^2 - \langle R(\gamma', X)\gamma', X\rangle dt = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b-\epsilon} |\nabla_t X|^2 - \langle R(\gamma', X)\gamma', X\rangle dt \\ &= \lim_{\epsilon \to 0^+} \left(\sum f^j \langle \nabla_t Y_j, X\rangle + \int_{a+\epsilon}^{b-\epsilon} \left|\sum {f^j}' Y_j\right|^2 dt \right) = \int_a^b \left|\sum {f^j}' Y_j\right|^2 dt \ge 0 \end{aligned}$$

Moreover, if $I(X, X) = 0$, then $\sum {f^j}' Y_j = 0$, so by linear independency, ${f^j}' = 0$, which means f^j are constant. Thus, $X \in J^\perp$.

Now, let $l = \dim \gamma_0 \cap J^\perp$ and $Y_1, \cdots, Y_l$ is a basis of $\gamma_0 \cap J^\perp$. Then, let $e_\mu = \nabla_t Y_\mu(a)$ for $\mu = 1, 2, \cdots, l$. Then, extend it to $\{e_1, \cdots, e_{n-1}\}$, which is a basis of $\gamma'(a)^\perp$. Then, let $Y_{l+1}, \cdots, Y_{n-1}$ be the Jacobi fields such that $Y_k(a) = 0$ and $\nabla_t Y_k(a) = e_k$. In other words, we actually not choose arbitrary linearly independent $Y_1, \cdots, Y_{n-1} \in \mathcal{J}^\perp$, but choose in this way. Then, if $a_{l+1}Y_{l+1}(b) + \cdots + a_{n-1}Y_{n-1}(b) = 0$, $a_{l+1}Y_{l+1} + \cdots + a_{n-1}Y_{n-1} \in \gamma_0 \cap \mathcal{J}^\perp$, which gives $a_{l+1} = \cdots = a_{n-1} = 0$ from linear independency of $\nabla_t Y_i(a)$'s and the fact that $Y_1, \cdots, Y_l$ is a basis of $\gamma_0 \cap \mathcal{J}^\perp$. Thus, $Y_{l+1}(b), \cdots, Y_{n-1}(b)$ are linearly independent. Moreover, since $Y_1(b) = Y_2(b) = \cdots = Y_l(b) = 0$, so $\nabla_t Y_1(b), \cdots, \nabla_t Y_l(b)$ are linearly independent since $|Y|^2 + |\nabla_t Y|^2 > 0$ for nonzero Jacobi field Y from **Theorem 97.** and $Y_1, \cdots, Y_l$ are linearly independent. Now, since $\langle Y_i(a), \nabla_t Y_j(a)\rangle - \langle Y_j(a), \nabla_t Y_i(a)\rangle = 0$ since $Y_i(a) = Y_j(a) = 0$ for every $i, j \in \{1, 2, \cdots, n-1\}$, so by **Theorem 98.**, $\langle \nabla_t Y_i(b), Y_j(b)\rangle = \langle \nabla_t Y_j(b), Y_i(b)\rangle = 0$ for $i \in \{1, 2, \cdots, l\}, j \in \{l+1, \cdots, n-1\}$. Thus, $\{\nabla_t Y_1(b), \cdots, \nabla_t Y_l(b)\} \perp \{Y_{l+1}(b), \cdots, Y_{n-1}(b)\}$, so $\{\nabla_t Y_1(b), \cdots, \nabla_t Y_l(b), Y_{l+1}(b), \cdots, Y_{n-1}(b)\}$ is a basis of $\gamma'(b)^\perp$. Now, if $\tau_{t_0,t}$ is a parallel transport from $T_{\gamma(t_0)}$ to $T_{\gamma(t)}$, by **Theorem 23.**, we get $X(t) = \tau_{t_0,t}(X(t_0) + (t - t_0)(\nabla_t X)(t_0)) + o(t - t_0)$. Then, let ξ^i be $(\nabla_t X)(b) = \sum_i^l \xi^i(\nabla_t Y_i)(b) + \sum_{i=l+1}^{n-1} \xi^i Y_i(b)$. Then, if $t < b$,

$$\begin{aligned}
X(t) &= \tau_{b,t}(X(b) + (t-b)(\nabla_t X)(b)) + o(b-t)\\
&= \tau_{b,t}((t-b)\sum_{i=1}^{l} \xi^i(\nabla_t Y_i)(b) + (t-b)\sum_{i=l+1}^{n-1} \xi^i Y_i(b)) + o(t-b)\\
&= \sum_{i=1}^{l} \xi^i Y_i(t) + (t-b)\sum_{i=l+1}^{n-1} \xi^i Y_i(t) + o(t-b)
\end{aligned}$$

since $X(b) = Y_1(b) = \cdots = Y_l(b) = 0$. Recall $Y_i(t) = \tau_{b,t}(Y_i(b) + (t-b)\nabla_t Y_i(b)) + o(t-b)$ also. Thus, $\lim_{t\to b^-} f^i(t) = \xi^i$ for $i = 1, \cdots, l$ and $\lim_{t\to b^-} f^i(t) = 0$ for $i = l+1, \cdots, n-1$. Thus, f^i are bounded to b, so bounded on $(a, b]$.

Lastly, for $t \to a^+$, choose η^i as $\nabla_t X(a) = \sum_i \eta^i e_i = \sum_i \eta^i(\nabla_t Y_i)(a)$. Then, similarily, $\lim_{t\to a^+} f^i(t) = \eta^i$, so bounded on $[a, b]$. □

Recall that if $X \in \gamma_0$, then $\frac{d^2L}{d\epsilon^2}(0) = I(X, X)$ for the variation of arc length L, where $\partial_\epsilon v(-, 0) = X$. Then, if there is a $X \in \gamma_0$ such that $I(X, X) < 0$, then it cannot be a length minimizing geodesic, since we can construct a variation $v(t, \epsilon) = \exp \epsilon X(t)$ of γ.

Theorem 106. For a unit speed geodesic γ defined on $[a, b]$, if $\gamma(a)$ has a conjugate point $\gamma(t_0)$ with $t_0 \in (a, b)$, there exists $X \in \gamma_0$ such that $I(X, X) < 0$. Thus, γ cannot be a length minimizing path over its first conjugate point.

Proof. Let $Y \in J^\perp$ such that $Y \not\equiv 0$ and $Y(a) = 0, Y(t_0) = 0$. Then, define

$$Y_1(t) = \begin{cases} Y(t) & t \in [a, t_0] \\ 0 & t \in [t_0, b] \end{cases}$$

Then, $Y_1 \in \gamma_0$ and $I(Y_1, Y_1) = (\mathcal{L}Y_1, Y_1) = 0$ since $\mathcal{L}Y_1 = 0$ on $[a, t_0]$, $Y_1 = 0$ on $[t_0, b]$. Since $Y(t_0) = 0$, $\nabla_t Y(t_0) \neq 0$. Then, let Z be a parallel transport along γ such that $Z(t_0) = -\nabla_t Y(t_0)$. Then, let $\varphi : [a, b] \to \mathbb{R}$ be a smooth map such that $\varphi(a) = \varphi(b) = 0, \varphi(t_0) = 1$. Lastly, define $X_\lambda = Y_1 + \lambda\varphi Z$. Then, $\partial_t\langle\varphi Z, \gamma'\rangle = \langle\varphi' Z + \varphi\nabla_t Z, \gamma'\rangle + \langle\varphi Z, \nabla_t\gamma'\rangle = \varphi'\langle Z, \gamma'\rangle$. Moreover, $\partial_t\langle Z, \gamma'\rangle = \langle\nabla_t Z, \gamma'\rangle + \langle Z, \nabla_t\gamma'\rangle = 0$ and $\langle Z(t_0), \gamma'(t_0)\rangle = -\langle\nabla_t Y(t_0), \gamma'(t_0)\rangle = 0$ since $Y \in J^\perp$. Thus, $\langle\varphi Z, \gamma'\rangle = 0$, which proves $X_\lambda \in \gamma_0$. Lastly,

$$\begin{aligned}
I(X_\lambda, X_\lambda) &= I(Y_1, Y_1) + 2\lambda I(Y_1, \varphi Z) + O(\lambda^2)\\
&= 2\lambda \int_a^b \langle\nabla_t Y_1, \nabla_t(\varphi Z)\rangle - \langle R(\gamma', Y_1)\gamma', \varphi Z\rangle dt + O(\lambda^2)\\
&= 2\lambda \int_a^{t_0} \langle\nabla_t Y, \nabla_t(\varphi Z)\rangle - \langle R(\gamma', Y)\gamma', \varphi Z\rangle dt + O(\lambda^2)\\
&= 2\lambda \int_a^{t_0} \langle\nabla_t Y, \nabla_t(\varphi Z)\rangle + \langle\mathcal{L}Y + \nabla_t^2 Y, \varphi Z\rangle dt + O(\lambda^2)\\
&= 2\lambda \int_a^{t_0} \partial_t\langle\nabla_t Y, \varphi Z\rangle dt + O(\lambda^2)\\
&= 2\lambda\langle\nabla_t Y, \varphi Z\rangle|_a^{t_0} + O(\lambda^2)\\
&= -2\lambda|\nabla_t Y(t_0)|^2 + O(\lambda^2)
\end{aligned}$$

Thus, for sufficiently small λ, $I(X_\lambda, X_\lambda) < 0$. □

2.12 Comparison Theorems

Theorem 107 (Bonnet-Myers)**.** Suppose $\gamma : [0, b] \to M$ is a unit speed geodesic and $Ric(\gamma', \gamma') \geq (n-1)\kappa > 0$ on $\gamma([0, b])$ for some κ. If $b \geq \frac{\pi}{\sqrt{\kappa}}$, then $\gamma((0, b])$ contains a point conjugate to $\gamma(0)$. Therefore, if M is complete with $\dim M \geq 2$ and there exists $\kappa > 0$ such that $Ric(\xi, \xi) \geq (n-1)\kappa|\xi|^2$ for any $\xi \in TM$, M is compact with diameter $\leq \frac{\pi}{\sqrt{\kappa}}$.

Proof. Choose an orthonormal basis $e_1, \cdots, e_{n-1}$ of $\gamma'(0)^\perp$. Let E_i be parallel transports of e_i's. Let $X_i(t) = (\sin\frac{\pi t}{b})E_i(t)$. Then, $X_i \in \gamma_0$. Moreover,

$$\sum_i I(X_i, X_i) \leq (n-1)\frac{b}{2}(\frac{\pi^2}{b^2} - \kappa)$$

by simple calculation using $\gamma', E_1, \cdots, E_{n-1}$ as an orthonormal basis to compute Ricci curvature tensor. Now, if $b \geq \frac{\pi}{\sqrt{\kappa}}$, then index form is not positive definite on γ_0, so γ has a conjugate point on $(a, b]$.

Now, for complete case, by Hopf-Rinow theorem, for any p, q, there exists a unit speed geodesic such that $\gamma(0) = p, \gamma(d(p,q)) = q$. Since it is a length minimizing, there is no conjugate point on $(0, d(p,q))$ which means $d(p,q) \leq \frac{\pi}{\sqrt{k}}$. Hence, M itself is a closed bounded subset of M, so is compact by **Corollary 58.** $\square$

Theorem 108 (Cartan-Hadamard)**.** For a unit speed geodesic $\gamma : [0,b] \to M$ satisfying $K \leq 0$ for every section on $\gamma([0,b])$, $\gamma((0,b])$ does not contain a conjugate point of $\gamma(0)$. Thus, if M is complete and every sectional curvature is nonpositive, then M has no conjugate pair.

Proof. For any $X \in \gamma_0$, $I(X,X) = \int_0^b |\nabla_t X|^2 - K(\gamma', X)|X|^2 dt \geq \int_0^b |\nabla_t X|^2 dt \geq 0$. Thus, it is true by **Theorem 106.**

Theorem 109 (Morse-Schönberg)**.** If $\delta > 0$ and $K \leq \delta$ on a unit speed geodesic $\gamma : [0,b] \to M$ such that $\gamma(b)$ is a conjugate point of $\gamma(0)$, then $b \geq \frac{\pi}{\sqrt{\delta}}$.

Proof. If $Y \in \gamma_0 \cap \mathcal{J}^\perp$, then $0 = I(Y,Y) \geq \int_0^b |\nabla_t Y|^2 - \delta |Y|^2 dt$. By Wirtinger's inequality, $\int_0^b |\nabla_t Y|^2 dt \geq (\pi^2/b^2) \int_0^b |Y|^2 dt$. Thus, done.

Remark. Wirtinger's inequality can be proved by Fourier series, or Rayleigh characterization of eigenvalue, which means $\Delta v + \lambda v = 0$ with Dirichlet condition, first eigenvalue $\lambda = \inf \frac{\int |\nabla v|^2}{\int v^2}$

Theorem 110 (Rauch)**.** For a real δ and a unit speed geodesic $\gamma : [0,b] \to M$ satisfying $K \leq \delta$ on γ, if $Y \in \mathcal{J}^\perp$, then $|Y|'' + \delta |Y| \geq 0$. Moreover, if ψ satisfying $\psi'' + \delta\psi = 0$, $\psi(0) = |Y|(0), \psi'(0) = |Y|'(0), \psi \neq 0$ on $(0,b)$, then $(|Y|/\psi)' \geq 0$, $|Y| \geq \psi$. Moreover, $(|Y|/\psi)' = 0$ at t_0 if and only if $K(Y, \gamma') = \delta$ on $[0, t_0]$, and there exists a parallel unit vector field E along γ such that $Y = \psi E$ on $[0, t_0]$.

Proof. At first, $|Y|' = \frac{\langle Y, \nabla_t Y\rangle}{|Y|}$. Use Jacobi equation, we gain

$$|Y|'' = \frac{|\nabla_t Y|^2 - \langle Y, R(\gamma', Y)\gamma'\rangle}{|Y|} - \frac{\langle Y, \nabla_t Y\rangle^2}{|Y|^3} \geq -\delta|Y| + \frac{|\nabla_t Y|^2 |Y|^2 - \langle Y, \nabla_t Y\rangle^2}{|Y|^3} \geq -\delta|Y|$$

Thus, the first part is done.

Now, let $F = |Y|'\psi - |Y|\psi'$. Then, $F(0) = 0, F' = |Y|'\psi - |Y|\psi' = |Y|''\psi - |Y|\psi'' \geq 0$. Thus, $(|Y|/\psi)' \geq 0$. Then, this gives, $|Y|/\psi \geq 1$, so $|Y| \geq \psi$. Lastly, if $(|Y|(t_0)/\psi(t_0))' = 0$, then $F(0) = F(t_0) = 0$ with $F' \geq 0$ implies $F = 0$ on $[0, t_0]$. Thus, $|Y| = \psi$ on $[0, t_0]$. Thus, we can write $Y = \psi E$ for some $|E| = 1$. Then, $\nabla_t Y = \psi' E + \psi \nabla_t E$. Then, we attain equality on $|Y|'' + \delta|Y| \geq 0$, so we attain equality of the Cauchy-Schwarz inequality. Thus, Y and $\nabla_t Y$ are linearly dependent. Since, $\psi \neq 0$ on $(0, t_0]$, $|E|^2 = 1$ is constant, so $E, \nabla_t E$ are orthogonal, which implies $\nabla_t E = 0$ on $[0, t_0]$. Inverse can be done easily by just calculation, so is done. $\square$

2.13 Jacobi fields and the Exponential map

Theorem 111. Let M be a Riemannian manifold with $\dim M \geq 2$. Fix $p \in M$ and $\xi \in \mathcal{T}M \cap T_pM$. Suppose Y is a Jacobi field along $\gamma(t) = \exp t\xi$ such that $Y(0) = 0, \nabla_t Y(0) = \eta \in T_pM$. Then, if $t\xi \in \mathcal{T}M$, $Y(t) = d(\exp_p)_{t\xi} I_{t\xi} t\eta$

Proof. For Z such that $Z(0) = \xi, Z'(0) = I_\xi \eta$, let $v(t, \epsilon) = \exp(tZ(\epsilon))$. By simple calculation, $Y(t) = \partial_\epsilon v(t, 0)$ is the Jacobi field satisfying theorem. It follows from the fact $\nabla_t \partial_\epsilon v = \nabla_\epsilon \partial_t v$. $\square$

Corollary 112. The kernel of $d(\exp p)_{t\xi}$ is isomorphic to the subspace of Jacobi fields along γ vanishing at p and $\exp \xi$.

Corollary 113 (Cartan-Hadamard Theorem)**.** The exponential map for a complete Riemannian manifold with nonpositve sectional curvature has maximal rank everywhere.

2.14 Riemann Normal Coordinates

Consider 2 dimensional case. In general, for a geodesic γ with a chart centered at $\gamma(0)$, even if $\gamma(1) = p$ is point (a,b), $\gamma(t) \neq (at, bt) = t \cdot (a,b)$. To do similar thing as this, we need some special coordinates.

Fix $p \in M$, and choose U as open, starlike respect to 0 in T_pM where exp is diffeomorphic between U and its image, which is an open set of M containing p. Then, if $e_1, \cdots, e_n$ is an orthonormal basis of T_pM, we may choose a chart $n : U = \exp \mathsf{U} \to \mathbb{R}^n$ as $n^j(q) = \langle (\exp|_{\mathsf{U}})^{-1}(q), e_j \rangle$. Then, if $v = \sum v^j e_j \in \mathsf{U}$, we get $n^j(\exp v) = v^j$. Then, for any $v \in T_pM$, $\gamma(t) = \exp tv$ we get $n(\gamma(t)) = tv, (n \circ \gamma)'(t) = v$ which means γ is a straight line in given chart.

Proposition 114. Suppose Y, Z are Jacobi fields such that $Y(0) = Z(0) = 0, \nabla_t Y(0) = \eta, \nabla_t Z(0) = \zeta$ along geodesic $\gamma(t) = \exp t\xi$ with $|\xi| = 1$. Then,

$$\langle Y, Z\rangle(t) = t^2\langle \eta, \zeta\rangle - \frac{t^4}{3}\langle R(\xi,\eta)\xi, \zeta\rangle + O(t^5)$$

Proof. To make equation simpler, use $'$ instead of ∇_t. It is enough to prove the case for the Jacobi fields $Y = Z = J$ such that $J'(0) = \eta$ with $|\eta| = 1$. We can write $J(t) = d(\exp_p)_{t\xi} I_{t\xi} t\eta$. First, $\langle J, J\rangle(0) = 0$. Now, $\langle J, J\rangle'(0) = 2\langle J, J'\rangle(0) = 0$. Then, $\langle J, J\rangle''(0) = 2\langle J'', J\rangle(0) + 2\langle J'(0), J'(0)\rangle = 2$. Now, $J''(0) = -R(\gamma', J)\gamma'|_0 = 0$. Thus, $\langle J, J\rangle'''(0) = 6\langle J'', J'\rangle(0) + 2\langle J''', J\rangle = 0$. Lastly, for any w,

$$\begin{aligned}\langle (R(\gamma', J)\gamma')', w\rangle(0) &= \langle R(\gamma', J)\gamma', w\rangle'(0) - \langle R(\gamma', J)\gamma', w'\rangle(0)\\ &= \langle R(\gamma', w)\gamma', J\rangle'(0)\\ &= \langle (R(\gamma', w)\gamma')', J\rangle(0) + \langle (R(\gamma', w)\gamma'), J'\rangle(0)\\ &= \langle (R(\gamma', J')\gamma'), w\rangle(0)\end{aligned}$$

which proves $(R(\gamma', J)\gamma')'(0) = R(\gamma', J')\gamma'(0)$. Then, $J'''(0) = -R(\gamma', J')\gamma'$. Use that, we can compute $\langle J, J, \rangle''''(0) = 8\langle J''', J'\rangle(0) + 6\langle J'', J''\rangle(0) + 2\langle J'''', J\rangle(0) = -8\langle R(\xi,\eta)\xi, \eta\rangle$. This proves, $\langle J, J\rangle(t) = t^2 - \frac{t^4}{3}\langle R(\xi,\eta)\xi,\eta\rangle + O(t^5)$. To gain the general case, use $\langle v, w\rangle = \frac{1}{4}(|v+w|^2 - |v-w|^2)$. □

Now, for the Riemannian normal coordinate based on $e_1, \cdots, e_n$, let Y_j be Jacobi fields such that $Y_j(0) = 0, \nabla_t Y_j(0) = e_j$. In other words, $Y_j(t) = d(\exp p)_{tv} I_{tv} t e_j$. Then, $t^{-1}Y_j(t) = d(\exp p)_{tv} I_{tv} e_j = \partial_j$.

Theorem 115. Under the Riemannian normal coordinate,

$$g_{jk}(\exp v) = \delta_{jk} - \frac{1}{3}\langle R(v, e_j)v, e_k\rangle + O(|v|^3)$$

Proof. Let $\xi = \frac{v}{|v|}$ and $|v| = t$. Then,

$$\begin{aligned} g_{jk}(\exp v) = g_{jk}(\exp t\xi) &= \langle t^{-1}Y_j, t^{-1}Y_k\rangle(t) = t^{-2}\langle Y_j, Y_k\rangle(t)\\ &= \langle e_j, e_k\rangle - \frac{t^2}{3}\langle R(\xi, e_j)\xi, e_k\rangle + O(t^3)\\ &= \delta_{jk} - \frac{1}{3}\langle R(v, e_j)v, e_k\rangle + O(|v|^3)\end{aligned}$$

□

Corollary 116. Under the Riemannian normal coordinate,

$$\det(g_{jk}(\exp(v)) = 1 - \frac{1}{3}Ric(v, v) + O(|V|^3)$$

Proof. Omit.

Recall **Theorem 100.**. If manifold has a constant secional curvature κ, then for any Jacobi field Y, there exist parallel vector fields E_1, E_2 such that $Y = (at+b)\gamma' + S_\kappa E_1 + C_\kappa E_2$, where $\gamma(t) = \exp t\xi$ for $|\xi| = 1$. Then, if we give condition $Y(0) = 0$, we can conclude $Y(t) = at\gamma' + S_\kappa E_1$. Then, if we use the Riemannian normal coordinate under $e_1, \cdots, e_n$, there exists a_j, E_j such that $Y_j(t) = a_j t\gamma'(t) + S_\kappa E_j(t)$. Then, since $\nabla_t E_j(0) = e_j$, we get $e_j = a_j\xi + E_j(0)$. Since E_j is orthogonal to γ', $\langle \xi, e_j\rangle = a_j|\xi|^2 = a_j$. Thus, $a_j = \xi^j$, so $E_j(0) = e_j - \xi^j\xi$. Lastly, we get $g_{jk}(\exp t\xi) = \xi^j\xi^k + \frac{S_\kappa^2(t)}{t^2}(\delta_{jk} - \xi^j\xi^k)$ from the direct calculation.

Theorem 117. For a Riemannian manifold M has a constant secional curvature κ, fix $p \in M$ and $\mathsf{U} \subseteq T_pM$ defining the Riemannian normal coordinate, we get

$$g_{jk}(\exp v) = \frac{v^j v^k}{|v|^2} + \frac{S_\kappa^2(|v|)}{|v|^2}(\delta_{jk} - \frac{v^j v^k}{|v|^2})$$

Proof. Use above result. □

Now, we do some formal calculation. If we consider $v = t\xi, |\xi| = 1$, we get $\langle \xi, d\xi\rangle = 0$ with $dv = t(d\xi) + (dt)\xi$. Also, $\sum v^j dv^j = \sum(v^j(dt\xi^j + td\xi^j)) = \sum(t{\xi^j}^2 dt + t^2\xi^j d\xi^j) = t|\xi|^2 dt + t^2\langle \xi, d\xi\rangle = tdt$. Under this result, let's compute $ds^2 = \sum g_{jk} dv_j dv_k$. First, $\sum_{j,k}\frac{v^j v^k}{|v|^2}dv_j dv_k = \frac{(tdt)^2}{t^2} = dt^2$. Moreover, $\sum \frac{S_\kappa^2(t)}{t^2}(\delta_{jk} - \frac{v^j v^k}{|v|^2})dv_j dv_k = \frac{S_\kappa^2(t)}{t^2}(\sum dv_j^2 - dt^2) = \frac{S_\kappa^2(t)}{t^2}(|dv|^2 - dt^2)$. Then, since $|dv|^2 = t^2|d\xi|^2 + dt^2|\xi|^2 = dt^2 + t^2|d\xi|^2$, we get $|dv|^2 - dt^2 = t^2|d\xi|^2$. Finally, we get $ds^2 = dt^2 + S_\kappa^2(t)|d\xi|^2$.

3 Riemannian Volume

Definition 118. Suppose $\xi \in T_pM$ and $|\xi| = 1$. Then $\mathsf{R}(t) : T_{\gamma_\xi(t)}M \to T_{\gamma_\xi(t)}M$ is defined as $\mathsf{R}(t)(\eta) = R(\gamma'_\xi(t), \eta)\gamma'_\xi(t)$, $\mathcal{R}(t) : T_pM \to T_pM$ is defined as $\mathcal{R}(t) = \tau_t^{-1} \circ \mathsf{R}(t) \circ \tau_t$ where τ_t is the parallel transport by γ_ξ from p to $\gamma_\xi(t)$. Remark that $\mathcal{R}(t)$ also be considered as $\mathcal{R}(t) : \xi^\perp \to \xi^\perp$. Then, $A(t, \xi)$ is a linear transform, in other word, matirx, which satisfy $A'' + \mathcal{R}(t)A = 0$ such that $A(0, \xi) = 0, A'(0, \xi) = I$.

Using such A, we can write a Jacobi field $Y \in \mathcal{J}^\perp$ satisfying $Y(0) = 0, \nabla_t Y(0) = \eta \in \xi^\perp$ as $Y(t) = \tau_t A(t, \xi)\eta$.

Definition 119. A **Conjugate locus of** $p \in M$ is a subset of $T_pM \cap \mathcal{T}M$ which is consisting of critical points of $\exp_p$. By **Corollary 112.**, it is equivalent to the collection of the vectors $t\xi \in T_pM \cap \mathcal{T}M$ such that $\det A(t, \xi) = 0$ where $|\xi| = 1$.

Definition 120. For $p \in M, \xi \in T_pM$ with $|\xi| = 1$, $c(\xi) = \sup\{t > 0 \mid t\xi \in \mathcal{T}M, d(p, \gamma_\xi(t)) = t\}$.

Remark. If $d(p, \gamma_\xi(t_1)) = t_1$ for some $t_1 > 0$, then $0 \le t_2 \le t_1$ implies $d(p, \gamma_\xi(t_2)) = t_2$. Moreover, if $0 \le t < c(\xi)$, then γ_ξ is the unique minimizing geodesic from p to $\gamma_\xi(t)$. If $|\eta| = 1$ and $\gamma_\eta(t) = \gamma_\xi(t)$, then for $T \in (t, c(\xi))$, we construct length minimizing broken geodesic, which actually being a geodesic, so $\eta = \xi$. Lastly, if $\gamma_\xi(T)$ is a conjugate point, then $c(\xi) \le T$.

Theorem 121. For a complete manifold M, if $c(\xi) < \infty$ for some $\xi \in \mathsf{S}_p$, $\gamma_\xi(c(\xi))$ is the first conjugate point along γ_ξ or there exist at least two minimizing geodesic connecting $\pi(\xi)$ and $\gamma_\xi(c(\xi))$.

Proof. First, since $c(\xi) < \infty$ and M is geodesically complete, $c(\xi)\xi \in \mathcal{T}M$. Consider decreasing t_j converges to $c(\xi)$. Let $d_j = d(p, \gamma_\xi(t_j))$. By Hopf-Rinow theorem, there exists $\eta_j \in \mathsf{S}_p$ such that $\gamma_\xi(t_j) = \gamma_{\eta_j}(d_j)$. Since $t_j > c(\xi)$, $d_j < t_j$. Now, there exists a converging subsequence η_{j_k}. Suppose it converges to ζ. If $\zeta = \xi$, it means $\exp_p$ is not one-to-one near $c(\xi)\xi$. Thus, $\exp_p$ has a critical point at $c(\xi)\xi$, which means $\gamma_\xi(c(\xi))$ is the first conjugate point. If $\zeta \ne \xi$, then $\gamma_\zeta(c(\xi)) = \gamma_\xi(c(\xi))$ where length by γ_ζ is also $c(\xi)$. Thus, there exist at least two minimizing geodesics. ☐

Definition 122. **Unit tangent bundle** $\mathsf{S}M = \{\xi \in TM \mid |\xi| = 1\}$.

Theorem 123. Function $c : \mathsf{S}M \to (0, \infty]$ is upper semicontinuous. If M is complete, then c is continuous.

Proof. Omit.

Definition 124. The **cut locus of** p **in** T_pM is $\mathsf{C}(p) = \{c(\xi)\xi \mid c(\xi) < \infty, \xi \in \mathsf{S}_pM\} \cap \mathcal{T}M$. Then, the **cut locus of** p **in** M is defined as $C_M(p) = \exp \mathsf{C}(p)$. Then, define $\mathsf{D}_p = \{t\xi \mid 0 \le t < c(\xi), \xi \in \mathsf{S}_p\}$ and $D_p^M = \exp \mathsf{D}_p$.

Theorem 125. D_p is the largest domain, starlike shape which $\exp_p|_{\mathsf{D}_p}$ is a diffeomorphism and moreover, $D_p^M = \exp(T_pM \cap \mathcal{T}M) \setminus C_M(p)$.

Proof. Omit.

Definition 126. For each $p \in M$, **injectivity radius of** p is defined as $\operatorname{inj} p = \inf\{c(\xi) \mid \xi \in \mathsf{S}_p\}$ and **injectivity radius of** M is defined as $\operatorname{inj} M = \inf\{\operatorname{inj} p \mid p \in M\}$.

Theorem 127. For $p \in M$ and $\xi \in \mathsf{S}_p$, if $\gamma_\xi(t_0)$ is a conjugate point of $\gamma_\xi(0)$ or there exists two minimizing geodesics connecting p and $\gamma_\xi(t_0)$, then $c(\xi) \le t_0$.

Proof. If conjugate point, then it cannot minimize over this point, so $c(\xi) \le t_0$. If there exists two minimizing geodesics, suppose σ is another such unit speed geodesic. Then, there is $\epsilon > 0$ small enought such that $\gamma_\xi(t_0 + \epsilon)$ is defined, and there exists unique geodesic τ connecting $\sigma(t_0 - \epsilon)$ and $\gamma_\xi(t_0 + \epsilon)$. Then, since σ, γ_ξ are distinct, so piecewise geodesic by σ, γ_ξ connecting them is not length minimizing. Thus, $\ell(\tau) < 2\epsilon$. Thus, $d(p, \gamma_\xi(t_0 + \epsilon)) \le t_0 - \epsilon + \ell(\tau) < t_0 + \epsilon$. Thus, $c(\xi) \le t_0$. ☐

Definition 128. A compact Riemannian manifold such that $C(p)$ reduces to a point is called as **Wiedersehen manifold**.

Theorem 129. Two dimensional Wiedersehen manifold is isometric to a sphere.

Remark that if q is a cut point, which means $\gamma_\xi(c(\xi))$, then p is a cut point of q along reversed γ. Also, by its definition, any point in D_p^M has the unique minimizing geodesic joined to p.

Theorem 130 (Klingenberg's Lemma). If M is a complete Riemannian manifold and $q \in C(p)$ such that $d(p,q) = d(p, C(q))$ where q is not conjugate to p along a minimizing geodesic, then q is the midpoint of a geodesic loop start and end at p. In particular, if M is a compact Riemannian manifold with sectional curvature bounded above by δ with $\delta > 0$, inj $M \le \min\{\frac{\pi}{\sqrt{\delta}}, \frac{\ell(M)}{2}\}$, where $\ell(M)$ is the length of the shortest simple closed geodesic in M.

Proof. By hypothesis, there exists two geodesics, γ_1, γ_2. To show it is a loop, enought to show that $\gamma_1'(L) = -\gamma_2'(L)$. Suppose not, choose a neightborhood U_1 of $\gamma_1'(0)$ and U_2 of $\gamma_2'(0)$ in S_p. Then, $\{\gamma_\xi(L) \mid \xi \in U_1\}$ and $\{\gamma_\eta(L) \mid \eta \in U_2\}$ intersect transversally, so there exists a sufficiently small ϵ such that $\{\gamma_\xi(L-\epsilon) \mid \xi \in U_1\} \cap \{\gamma_\eta(L-\epsilon) \mid \eta \in U_2\} \neq \emptyset$ which is contradiction to q is the closest to $C(p)$. □

Theorem 131. Distance function $d_p : M \to \mathbb{R}$ defined as $d_p(q) = d(p,q)$ is smooth on $M \setminus \{C_M(p) \cup \{p\}\}$.

Proof. Sketch of proof. Choose A as $\exp|_A : A \to M \setminus (C_M(p) \cup \{p\})$ is diffeomorphic. Then, $d_p(q) = |\exp_p^{-1}(q)| = |v|$. Then, for any vector $X \in T_qM$, choose a smooth curve σ in $M \setminus (C_M(p) \cup \{p\})$ such that $\sigma(0) = q$ and $\sigma'(0) = X$. Then, consider geodesic variation γ_s such that unique minizing geodesic from p to $\sigma(s)$. Then compute $\frac{dL}{ds}(0) = X(d_p) = \langle X, \gamma'(d_p(q))\rangle$ where $\gamma = \gamma_0$. □

3.1 Riemannian measure

How to define integral on Riemannian manifold? First, consider a compact subset K of (M,g) such that included in a domain U of a chart $x : U \to \mathbb{R}$. Then, $x(K)$ is measurable. Then, we may define

$$\mathrm{Vol}(K) = \int_{x(K)} \sqrt{\det g^x} \circ x^{-1} dx_1 \cdots dx_n$$

Use $g^x = J^T g^y J$ as before, we can conclude this definition does not depends for the choice of charts by simple calculation.

Then, for general case, we may define integral by a partition of unity, and it is easy to prove that integral does not depends for the choice of the partition of unity. Then, if ρ_α is a partition of unity, we might denote $dV = \sum_\alpha \rho_\alpha \sqrt{\det g^{x_\alpha}} \circ x_\alpha^{-1} dx_1 \cdots dx_n$ just in formally, so we might use notation $\int f dV$.

3.2 Volume comparison theorems

Theorem 132 (Günther-Bishop Theorem). For a unit speed geodesic γ_ξ such that sectional curvatures along this curve is bounded above by δ,

$$\frac{(\det A(t,\xi))'}{\det A(t,\xi)} \ge (n-1)\frac{S_\delta'}{S_\delta}$$

on $(0, \frac{\pi}{\sqrt{\delta}})$ and $\det A \ge S_\delta^{n-1}$ on $(0, \frac{\pi}{\sqrt{\delta}}]$. If $\delta \le 0$, then both are true for $(0, \infty)$.

Proof. Let $B = A^*A$. Then, $\det B = (\det A)^2$. Then, $\ln \det B = 2 \ln \det A$, so $\frac{(\det A)'}{\det A} = \frac{(\det B)'}{\det B} = \frac{1}{2}\mathrm{tr}(B'B^{-1})$. Now suppose $e_1, \cdots, e_{n-1}$ is a orthonormal basis of $\xi^\perp$, composed by eigenvectors of B, which is symmetric. Let $\eta_i = Ae_i$. Then, $Be_i = \lambda_i e_i$ where $\lambda_i = \langle Be_i, e_i\rangle = \langle A^*Ae_i, e_i\rangle = \langle Ae_i, Ae_i\rangle = \langle \eta_i, \eta_i\rangle$. Then, $\lambda_i' = 2\langle \eta_i', \eta_i\rangle$. Thus, $\frac{1}{2}\mathrm{tr}B'B^{-1} = \sum \frac{\lambda_i'}{2\lambda_i} = \sum \frac{\langle \eta_i', \eta_i\rangle}{\langle \eta_i, \eta_i\rangle}$. Then, by **Theorem 110.** with $\psi = cS_\delta$, $Y = \eta_i$, $\frac{|Y|'}{|Y|} = \frac{(\sqrt{\langle \eta_i, \eta_i\rangle})'}{\sqrt{\langle \eta_i, \eta_i\rangle}} \ge \frac{\psi'}{\psi}$ gives result. Second inequality is just integrating. □

For second inequlity, LHS is a volume element of M, which means $dV(\exp t\xi) = \det A(t,\xi) dt d\mu_p(\xi)$ in some sence, where this can be done by some calculation. RHS is just a volume element of a constant sectional curvature manifold with δ. From formula $ds^2 = dt^2 + S_\delta^2(t)|d\xi|^2$ for constant sectional curvature, it is easy to show that volume element of such manifold is $\sqrt{\det g} = \sqrt{1 \cdots (S_\delta^2)^{n-1}} = S_\delta^{n-1}$. Also, by some more calculation, we can prove that volume of S^{n-1} is $c_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$, and volume of a Ball B^n is $w_n = \frac{c_{n-1}}{n}$.

Theorem 133. If sectional curvature is bounded above by δ, then volumn of a ball in M, $V(x,r)$, satisfies $V(x,r) \ge V_\delta(x,r)$ where $V_\delta(x,r)$ is a volume of a disk in a manifold with constant sectional curvature δ for every $r \le \min\{\mathrm{inj}\ x, \pi/\sqrt{\delta}\}$.

Proof. Omit.

Theorem 134 (Bishop). If the Ricci curvature is bounded below by $(n-1)\kappa$ along unit speed geodesic γ_ξ until its first conjugate point,

$$\frac{(\det A(t,\xi))'}{\det A(t,\xi)} \le (n-1)\frac{S_\kappa'}{S_\kappa}$$

from 0 to first conjugate point. Also, $\det A \le S_\kappa^{n-1}$ is also true. Moreover, equality holds to t_0 if and only if $A = S_\kappa I$, $\mathcal{R} = \kappa I$ for every $t \in (0, t_0]$.

Proof. Define $\psi = (n-1)\frac{S'_\kappa}{S_\kappa} = (n-1)CT_\kappa$. Then, $\psi' < 0$, $\psi' + \frac{\psi^2}{n-1} + (n-1)\kappa = 0$ which is called as the Riccati equation. Now, consider $\phi = \mathrm{tr}U = trA'A^{-1} = \frac{(\det A)'}{\det A}$. Then, for Wronskian $W(L,T) = (L')^*T - L^*T'$, we get $W(A,A) = 0$ from equation $A'' + \mathcal{R}A = 0$ and the fact that $\mathcal{R}$ is self-adjoint. Then, $U^* - U = (A^{-1})^*W(A,A)A^{-1} = 0$ which means U is self-adjoint. Moreover, $U' + U^2 + \mathcal{R} = 0$, which is the matrix Riccati equation, and then, $(\mathrm{tr}U)' + \mathrm{tr}U^2 + \mathrm{tr}\mathcal{R} = 0$. Then, by Cauchy-Schwarz inequality, $\mathrm{tr}U^2 \geq \frac{(\mathrm{tr}U)^2}{n-1}$ which gives $\phi' + \frac{\phi^2}{n-1} + (n-1)\kappa \leq 0$. Then, integrate $\frac{-\phi'}{\frac{\phi^2}{n-1} + (n-1)\kappa} \geq 1$ gives $CT_\kappa^{-1}\frac{\phi}{n-1} \geq s$, finially conclude $\psi \geq \phi$. Specific calculation are omitted. $\square$

Lemma 135 (Gromov)**.** Suppose f, g are positive integrable, f/g is decreasing, then $\int_0^r f / \int_0^r g$ also decreases.

Proof. For $R > r > 0$, $\int_0^r f \int_0^R g = \int_0^r f \int_0^r g + \int_0^r f \int_r^R g \geq \int_0^r f \int_0^r g + \int_0^r \frac{f(r)}{g(r)} g \int_r^R g = \int_0^r f \int_0^r g + \int_0^r g \int_r^R \frac{f(r)}{g(r)} g \geq \int_0^r f \int_0^r g + \int_0^r g \int_r^R f = \int_0^r g \int_0^R f$. $\square$

For $\mathsf{D}_x(r) = \frac{1}{r}\mathsf{D}_x \cap \mathsf{S}(x,r)$, define lower area $Q(x,r) = \int_{\mathsf{D}_x(r)} \det A(r,\xi) d\mu_x(\mathsf{S}_x)$. Let $A_\kappa(r) = c_{n-1}S_\kappa^{n-1}(r)$.

Proposition 136. If the Ricci curvature is bounded below by $(n-1)\kappa$, then $\frac{Q(x,r)}{A_\kappa(r)}$ decreases with respect to r.

Proof. Simple consequence of the Bishop theorem, which gives $\frac{\det A}{S_\kappa^{n-1}}$ decreasing. $\square$

Theorem 137 (Gromov)**.** If the Ricci curvature is bounded below by $(n-1)\kappa$, then $\frac{V(x,r)}{V_\kappa(r)}$ decreases with respect to r.

Proof. Since $V(x,r) = \int_0^r Q(x,s)ds, V_\kappa(r) = \int_0^r A_\kappa(s)ds$, it just follows by **Lemma 135.**. $\square$

4 Riemannian Covering

Definition 138. For two connected topological manifolds $\widetilde{M}, M$, $\psi : \widetilde{M} \to M$ is a **covering map** if for every $p \in M$, there is a connected open neighborhood U of p such that each component of $\psi^{-1}(U)$ is homeomorphic to U.

Theorem 139 (Unique Lifting Theorem)**.** For any path $w : [0,b] \to M$ such that $w(0) = p$ with a covering map $\psi : \widetilde{M} \to M$, for every $\widetilde{p} \in \psi^{-1}(p)$, there exists a unique path $\widetilde{w} : [0,b] \to \widetilde{M}$ such that $w = \psi \circ \widetilde{w}$ and $\widetilde{w}(0) = \widetilde{p}$.

Definition 140. A covering map $\psi : \widetilde{M} \to M$ between two differentiable manifolds is a **differentiable covering** if it is a differentiable covering map with maximal rank on $\widetilde{M}$.

Definition 141. A covering map between two Riemannian manifolds is a **Riemannian covering** if it is a differentiable covering and a local isometry.

Proposition 142. If $\psi : \widetilde{M} \to M$ is a Riemannian covering, then M is complete if and only if $\widetilde{M}$ is complete.

Proof. Since ψ is a local isometry, $\exp(d\psi_{\pi(\xi)}\xi) = \psi(\exp \xi)$ for every $\xi \in \mathcal{T}\widetilde{M}$. In other words, geodesic preserved. Now, if $\widetilde{M}$ is complete, then it is geodesically complete. Now, for every $\xi \in T_pM$, since differentiable covering, there exists a $\widetilde{\xi} \in T_{\widetilde{p}}\widetilde{M}$ such that $d\psi_{\widetilde{p}}\widetilde{\xi} = \xi$. Then, there exists a geodesic $\gamma : \mathbb{R} \to \widetilde{M}$ such that $\gamma'(0) = \widetilde{\xi}$. Now, $\psi \circ \gamma$ is a geodesic of M such that $(\psi \circ \gamma)'(0) = \xi$, which proves M is also geodesically complete, so is complete.

Also, if M is complete, then it can be done easily that lifted geodesic by **Theorem 139.** is a geodesic. Thus, $\widetilde{M}$ is geodesically complete, so is complete. $\square$

Theorem 143 (Myers)**.** For any complete Riemannian manifold M with Ricci curvature is bounded below by positive constant, M is compact and if $\psi : \widetilde{M} \to M$ is a Riemannian covering, then $\widetilde{M}$ is compact.

Proof. Omit.

Remark. There is a sectional curvature version theorem of this theorem.

Proposition 144. If $\phi : X \to Y$ is a local isometry, then for every $p \in X$, there exists an $\epsilon > 0$ such that $\phi|_{B(p,\epsilon)} B(p,\epsilon) \to B(\phi(p),\epsilon)$ is an isometry.

Proof. Let $\epsilon_p > 0$ satisfying $\exp|_{\mathsf{B}(p,\epsilon_p)} : \mathsf{B}(p, \epsilon_p) \to B(p, \epsilon_p)$ be an isometry. Similarily, define $\epsilon_{\phi(p)}$. Then, choose $\epsilon = \min\{\epsilon_p, \epsilon_{\phi(p)}\}$. Then, $d\phi_p|_{\mathsf{B}(p,\epsilon)} : \mathsf{B}(p, \epsilon) \to \mathsf{B}(\phi(p), \epsilon)$ is an isometry. Thus, $\phi|_{B(p,\epsilon)} = \exp_{\phi(p)} \circ d\phi_p \circ \exp|^{-1}_{\mathsf{B}(p,\epsilon)}$ is an isometry. Equality is from preserving geodesic property. □

Theorem 145. If $\widetilde{M}$ is a connected complete Riemannian manifold and $\psi : \widetilde{M} \to M$ is a surjective local isometry, then ψ is a covering.

Proof. First, let $\gamma : [0, T_0] \to M$ be a segment of a geodesic. Choose $\widetilde{p} \in \widetilde{M}$ such that $\psi(\widetilde{p}) = \gamma(0)$. First, note that by above proposition, there exists a unique lifted geodesic defined on $[0, \epsilon)$. Now, let $T = \sup\{\tau | \gamma|_{[0,\tau]}$ has a lifting starting at $\widetilde{p}\}$. In other words, for some $\widetilde{\xi} \in T\widetilde{M}$ such that $d\phi_{\widetilde{p}}\widetilde{\xi} = \gamma'(0)$, $\psi(\gamma_{\widetilde{\xi}}(t)) = \gamma(t)$ for $t \in [0, T)$. Since complete, $\gamma_{\widetilde{\xi}}(T)$ is defined, and $\psi(\gamma_{\widetilde{\xi}}(T)) = \lim_{t \to T^-} \gamma(t)$. Now, again, use small ϵ, $T < T_0$ implies lift can be extended, which is contradiction. Thus, $T = T_0$, so every segmenet of a geodesic can be lifted to a geodesic.

Now, fix $p \in M$, and choose $\epsilon > 0$ as $\exp|_{\mathsf{B}(p,\epsilon)}$ is a diffeomorphism onto $B(p, \epsilon)$. First, choose $\widetilde{q} \in \bigcup_{\widetilde{p} \in \psi^{-1}(p)} B(\widetilde{p}, \epsilon)$. Then, there is a $\widetilde{p} \in \psi^{-1}(p)$ and a path $\widetilde{w}$ joining $\widetilde{p}$ and $\widetilde{q}$ with $\ell(\widetilde{w}) < \epsilon$. Then, $w = \psi \circ \widetilde{w}$ is a path joining p and $\psi(\widetilde{q})$ with $\ell(w) < \epsilon$, so $\psi(\widetilde{q}) \in B(p, \epsilon)$. Thus, $\bigcup_{\widetilde{p} \in \psi^{-1}(p)} B(\widetilde{p}, \epsilon) \subseteq \psi^{-1}(B(p, \epsilon))$.

Then, for $\widetilde{q} \in \psi^{-1}(B(p, \epsilon))$, let $q = \psi(\widetilde{q})$. By choice of ϵ, there is a geodesic $\gamma : [0, T] \to M$ joining p and q with $\ell(\gamma) < \epsilon$. Then, there is a lift $\widetilde{\gamma}$ defined on $\widetilde{M}$ such that start at $\widetilde{q}$ with $\ell(\widetilde{\gamma}) < \epsilon$. Then, since lift, $\widetilde{q} \in \bigcup_{\widetilde{p} \in \psi^{-1}(p)} B(\widetilde{p}, \epsilon)$, so $\bigcup_{\widetilde{p} \in \psi^{-1}(p)} B(\widetilde{p}, \epsilon) = \psi^{-1}(B(p, \epsilon))$. Note that this is also true for every smaller ϵ. Now, if $\widetilde{p}_1, \widetilde{p}_2 \in \psi^{-1}(p)$ with $\widetilde{p}_1 \neq \widetilde{p}_2$, since diffeomorphic, $d(\widetilde{p}_1, \widetilde{p}_2) \geq \epsilon$. Then, $B(\widetilde{p}_1, \epsilon/3) \cap B(\widetilde{p}_2, \epsilon/3) = \emptyset$, so $B(p, \epsilon/3)$ is a desired neighborhood proving covering map. □

Theorem 146 (Cartan-Hadamard)**.** For a complete Riemannian manifold M with every sectional curvature is non-positive, $\exp_p : T_pM \to M$ is a covering map.

Proof. For metric g, equip metric $(\exp_p)^*g$ on T_pM. This is possible from **Corollary 113.** to define pull-back universally. Then, geodesics on T_pM are straight lines from origin, thus geodesically complete, so is complete. Now, by above theorem, proof is done. □

Remark. If M is a complete Riemannian manifold with every sectional curvature is non-positive and simply connected, then $\exp_p$ is a covering map and has only one sheet, so $\exp_p$ is a diffeomorphism. In other words, M is diffeomorphic to $\mathbb{R}^n$.

4.1 Orientability

Definition 147. For a manifold M, two charts $(U, \phi), (V, \psi)$ with $U \cap V \neq \emptyset$ have **same orientation** if $\psi \circ \phi^{-1}$ has positive determinanat everywhere.

Definition 148. A set of charts is **oriented** if any pair of overlapping charts have same orientation.

Definition 149. A manifold M is **orientable** if it has a oriented atlas. Manifold is oriented if there is a given oriented atlas. Any basis of T_pM for $p \in M$ is a **positive basis** if it has same orientation with given oriented atlas, and is **negative basis** if not.

Definition 150. For a vector space V, two basis $\{e_1, \cdots, e_n\}$, $\{v_1, \cdots, v_n\}$ have **same orientation** if $e^1 \wedge \cdots \wedge e^n(v_1, \cdots, v_n) > 0$ where e^i are dual basis of $e_1, \cdots, e_n$. Two basis has **opposite orientation** if $e^1 \wedge \cdots \wedge e^n(v_1, \cdots, v_n) < 0$.

Proposition 151. Two charts $(U, x), (V, y)$ have same orientation if $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}$ have same orientation everywhere.

Definition 152. A **volume form** of a n-dimensional manifold is a n-differential form which never vanishes.

Proposition 153. Manifold is orientable if and only if it has a volume form.

Definition 154. For a local diffeomorphism $F : M \to N$ between oriented manifolds, F **preserves the orientation** if for every $p \in M$, image by dF_p of a positive basis of T_pM is a positive basis of $T_{F(p)}M$. F **reverses the orientation** if for every $p \in M$, image by dF_p of a positive basis of T_pM is a negative basis.

Proposition 155. For a connected non-orientable manifold M, there is a smooth covering $\pi : \widetilde{M} \to M$ with two sheets such that $\widetilde{M}$ is connected and orientable. The automorphism, also called as deck transformation, group of the coverings is isomorphic to $\mathbb{Z}_2$. Moreover, if $F : \widetilde{M} \to \widetilde{M}$ is a automorphism which is not identity, then F reverses the orientation of $\widetilde{M}$.

Corollary 156. Every simply connected manifold is orientable

Lemma 157. For $A \in O(n-1)$ with $\det A = (-1)^n$, 1 is an eigenvalue of A. i.e., there exists v such that $Av = v$.

Proof. Recall that every eigenvalue of A is 1, -1 or a nonreal complex number which always appear as a conjugate pair. Thus, $\det A = 1$ if and only if -1 has even multiplicity, and $\det A = -1$ if and only if -1 has odd multiplicity. Then, just using parity argument, 1 is an eigenvalue of A. □

Lemma 158. The parallel transport along any curve preserves the orientation.

Proof. Supppose curve is $\sigma : [a,b] \to M$ and $E_1, \cdots, E_n$ is a basis of $T_{\sigma(a)}M$. Let $E_1(t), \cdots, E_n(t)$ is a extended vector fields along σ by parallel transportation. Now, consider a volume form ν of M. Then, $\nu_{\sigma(t)}(E_1(t), \cdots, E_n(t))$ is nonvanishing, so its parity is preserved. In other words, orientation preserved. □

Theorem 159 (Weinstein-Synge)**.** For an isometry $F : M \to M$ with a compact orientable M with positive sectional curvatures, if $n = \dim M$ is even and F preserves orientation, or n is odd and F reverses orientation, then F has a fixed point.

Proof. Suppose F has no fixed point. Then, since M is compact, function $p \mapsto d(p, F(p))$ has a minimum which is not zero. Let p be a minimum point. Then, since M is compact, so M is complete. Thus, there is a minimizing unit speed geodesic $\sigma : [0,l] \to M$ where $l = d(p, F(p))$ such that $\sigma(0) = p, \sigma(l) = F(p)$. At first, we will prove $dF_p(\sigma'(0)) = \sigma'(l)$.

Remark that $d(p, F(p)) \leq d(\sigma(t), F(\sigma(t))) \leq d(\sigma(t), F(p)) + d(F(p), F(\sigma(t))) = d(\sigma(t), F(p)) + d(p, \sigma(t)) = d(p, F(p))$ for any $t \in [0,l]$. Last equality is from that any part of the length minimizing unit speed curve is also length minimizing. Then, $d(\sigma(t), F(\sigma(t))) = d(\sigma(t), F(p)) + d(F(p), F(\sigma(t)))$, so concating $\sigma|_{[t,l]}$ and $F \circ \sigma|[0,t]$ is a length minimizing curve, which means it is also a geodesic. By uniqueness of the geodesic, we can conclude concating σ and $F \circ \sigma$ is a geodesic. In other words, $F(\sigma(t)) = \sigma(t+l)$ when we extend a geodesic σ. Hence, $(F \circ \sigma)'(0) = \sigma'(l)$, which is equivalent to $dF_p(\sigma'(0)) = \sigma'(l)$.

Now, let τ be a parallel transport from p to $F(p)$ along σ. Let $\widetilde{A} = \tau^{-1} \circ dF_p : T_pM \to T_pM$. Then, $\widetilde{A}$ is an isometry, so $\widetilde{A} \in O(n)$ and if F preserve orientation, then $\widetilde{A}$ preserve orientation, F reverses orientation, then $\widetilde{A}$ reverses orientation. In other words, $\det \widetilde{A} = (-1)^n$. Now, $\widetilde{A}(\sigma'(0)) = \tau^{-1}(\sigma'(l)) = \sigma'(0)$, so if we define $W = \sigma'(0)^\perp$, then $A = \widetilde{A}|_W : W \to W$ and $\det A = (-1)^n$. Thus, $A \in O(n-1)$ and $\det A = (-1)^n$, so there is a vector $E_1 \in W$ such that $AE_1 = E_1$. Now, let $E_1(t)$ is extended E_1 by parallel transport. Now, consider a geodesic $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p, \gamma'(0) = E_1$. Then, $(F \circ \gamma)'(0) = dF_p(E_1(0)) = dF_p(E_1) = \tau \circ \widetilde{A}(E_1) = \tau(E_1) = E_1(l)$.

Then, consider a variation of σ, $\Sigma(s,t) = \exp_{\sigma(t)} sE_1(t)$ which satisfies $V(t) = \partial_s \Sigma(0,t) = E_1(t)$. Then, Σ along s is a geodesic, so $\nabla_s \partial_s \Sigma = 0$. Thus, the 2nd variation of the length is

$$\begin{aligned} L''(0) &= \langle \nabla_s \partial_s \Sigma, \sigma' \rangle|_0^l + \int_0^l |\nabla_t E_1|^2 - \langle R(\sigma', E_1)\sigma', E_1 \rangle dt \\ &= -\int_0^l \langle R(\sigma', E_1)\sigma', E_1 \rangle dt = -\int_0^l K(\sigma', E_1) dt < 0 \end{aligned}$$

Thus, there is s_0 such that $\ell(\Sigma(s_0, -)) < \ell(\Sigma(0, -)) = l$.

Lastly, from that $(F \circ \gamma)'(0) = E_1(l)$, since $\Sigma(s,l)$ is a geodesic such that $\partial_s \Sigma(0,l) = E_1(l)$, by uniqueness of the geodesic, $\Sigma(s,l) = F \circ \gamma(s)$. The definition of γ is nothing but $\gamma(s) = \Sigma(s,0)$. Thus, $\Sigma(s_0, -)$ is a curve connecting $\Sigma(s_0, 0) = \gamma(s_0)$ and $\Sigma(s_0, l) = F(\gamma(s_0))$ which is contradiction. □

Following theorem needs some knowledge of algebraic topology.

Theorem 160 (Synge)**.** For a compact Riemannian manifold M with dimension n which has positive sectional curvatures,

1. If n is even and M is orientable, then M is simply connected.

2. If n is even and M is non-orientable, then $\pi_1(M) \cong \mathbb{Z}_2$.

3. If n is odd, then M is orientable.

Proof. First, assume n is even and M is orientable. Let $\pi : \widetilde{M} \to M$ be a univeral covering. Then, we may give a pull-back metric from M and an orientation matches with M on $\widetilde{M}$. In other words, volume form of $\widetilde{M}$ is given as the pull-back of the volume form of M. Then, by **Theorem 143.** of sectional curvature version, $\widetilde{M}$ is compact. Now, let $F : \widetilde{M} \to \widetilde{M}$ be an automorphism of the covering, which means $\pi \circ F = \pi$. Then, F is an isometry which proeserves orientation. Thus, F has a fixed point, so F is an identity, which means the group of automorphisms of the covering is the trivial group, which proves M is simply connected.

Now, if n is even and M is orientable, then consider natural 2-sheet covering $\pi : \widetilde{M} \to M$. Then, by part 1, $\widetilde{M}$ is simply connected, so is a universal covering. Thus, $\pi_1(M) \cong \mathbb{Z}_2$.

Lastly, when n is odd, suppose M is not orientable. Then, consider natural 2-sheet covering $\pi : \widetilde{M} \to M$ and a non-trivial automorphism F of covering. From **Proposition 155.**, F reverses the orientation, so F has a fixed point. Then, F must be the identity, so is contradiction. □

Remark. For $\mathbb{R}P^n$, $\pi : S^n \to \mathbb{R}P^n$ is a natural universal covering, with unique nontrivial automorphism $A(p) = -p$, which is the antipodal map, an isometry of S^n. Thus, $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ always, so $\mathbb{R}P^n$ is orientable if and only if n is odd. Note that $\mathbb{R}P^n$ is a compact Riemannian manifold with positive sectional curvatures.